Describing shell-crossing structure in cosmological Vlasov-Poisson dynamics

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based on arXiv:1805.08787

with Atsushi Taruya (YITP) and Stéphane Colombi (IAP)

1 Cold Dark Matter

Cold Dark Matter

Galaxy rotation curve, bullet cluster, LSS, CMB, …

- initially **negligible velocity dispersion**
- single-stream flow (= fluid approximation is valid)

Fate of CDM

➢ **Shell-crossing**

Breakdown of single-stream flow at **shell-crossing**

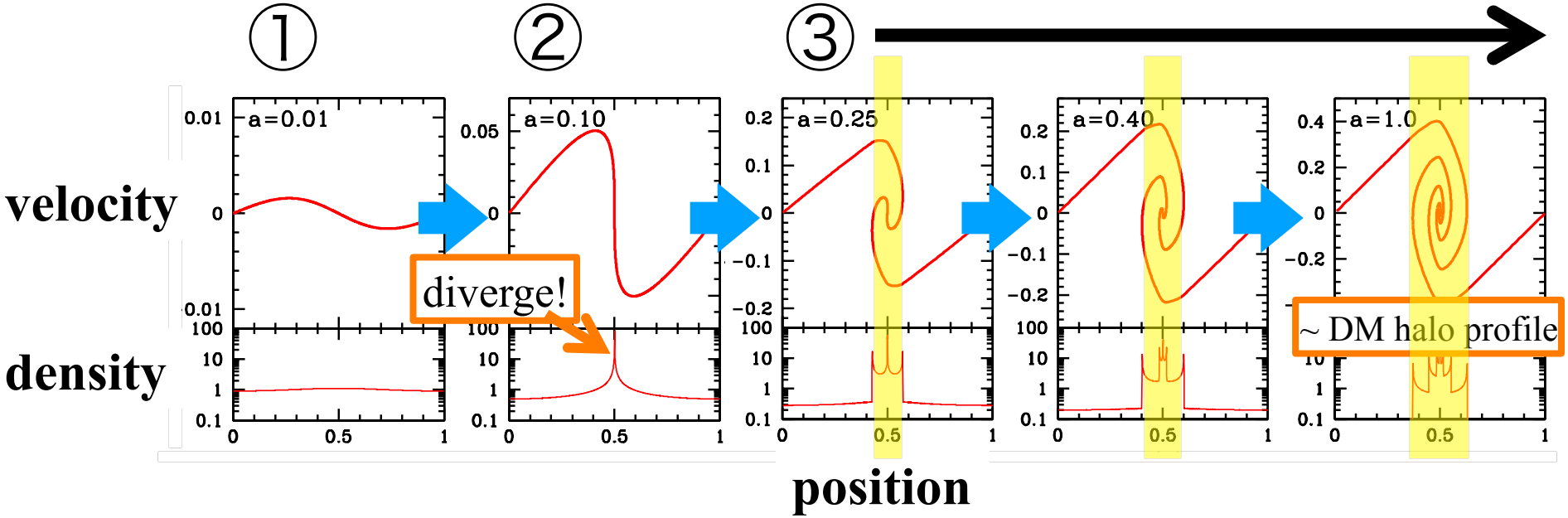
- singular density field
- fluid approximation is invalid

➢ **Multi-stream flow**

- On small scales, multi-stream flow dominates.
- Splashback radius

1.1 Phase-space structure

Phase-space in 1D is useful to intuitively understand shell-crossing.



① we set a small density peak at the origin (single-stream flow)
② **density peak diverges** or **shell-crossing occurs** at the origin
③ multi-stream flow appears

1.2 Summary of Introduction

Background

- **shell-crossing** is inevitable phenomenon if dark matter is **cold**.
- in 1D case, there is **an exact solution** up to shell-crossing
- **little knowledge about 3D** shell-crossing.
- high-resolution 6D Vlasov-Poisson simulation is developed.

Goal

- Describing **shell-crossing**, especially, by perturbation theory
- Convergence study of perturbation theory

Method

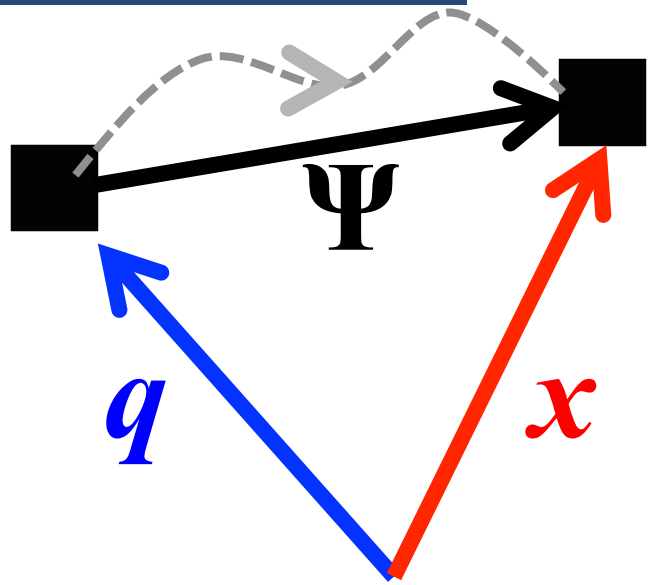
Lagrangian perturbation theory

2. Lagrangian Perturbation Theory

Eulerian coordinate **Displacement field**

$$\boldsymbol{x}(\boldsymbol{q},t) = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q},t)$$

Lagrangian coordinate (initial position)



$$\textbf{LPT} \quad \boldsymbol{\Psi}(\boldsymbol{q},t) = \sum_{n=1}^{\infty} \boldsymbol{\Psi}^{(n)}(\boldsymbol{q},t)$$

Basic equation (pressureless, Newtonian gravity):

$$\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}} = -\frac{1}{a^2}\nabla_x\phi$$

$$\nabla_x^2\phi(\boldsymbol{x},t) = 4\pi G\bar{\rho}(t)a^2(t)\delta(\boldsymbol{x},t)$$

or, equivalently:

Longitudinal part: $\nabla_x \cdot \left[\ddot{\boldsymbol{\Psi}} + 2H\dot{\boldsymbol{\Psi}}\right] = -4\pi G\bar{\rho}_{\rm m}\delta$

Transverse part: $\nabla_x \times \left[\ddot{\boldsymbol{\Psi}} + 2H\dot{\boldsymbol{\Psi}}\right] = 0$

3. Initial condition and simulation

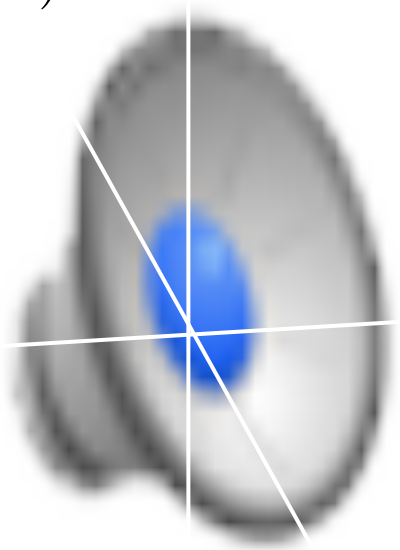
ColDICE: 6D Vlasov-Poisson solver

T.Sousbie and S.Colombi [1509.07720]

$$\frac{\partial f(t,\boldsymbol{x},\boldsymbol{u})}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} f(t,\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla} f(t,\boldsymbol{x},\boldsymbol{u}) = 0$$

$$\nabla^2\phi = 4\pi G\int \mathrm{d}^3\boldsymbol{u} f(t,\boldsymbol{x},\boldsymbol{u})$$

https://vlasix.org/

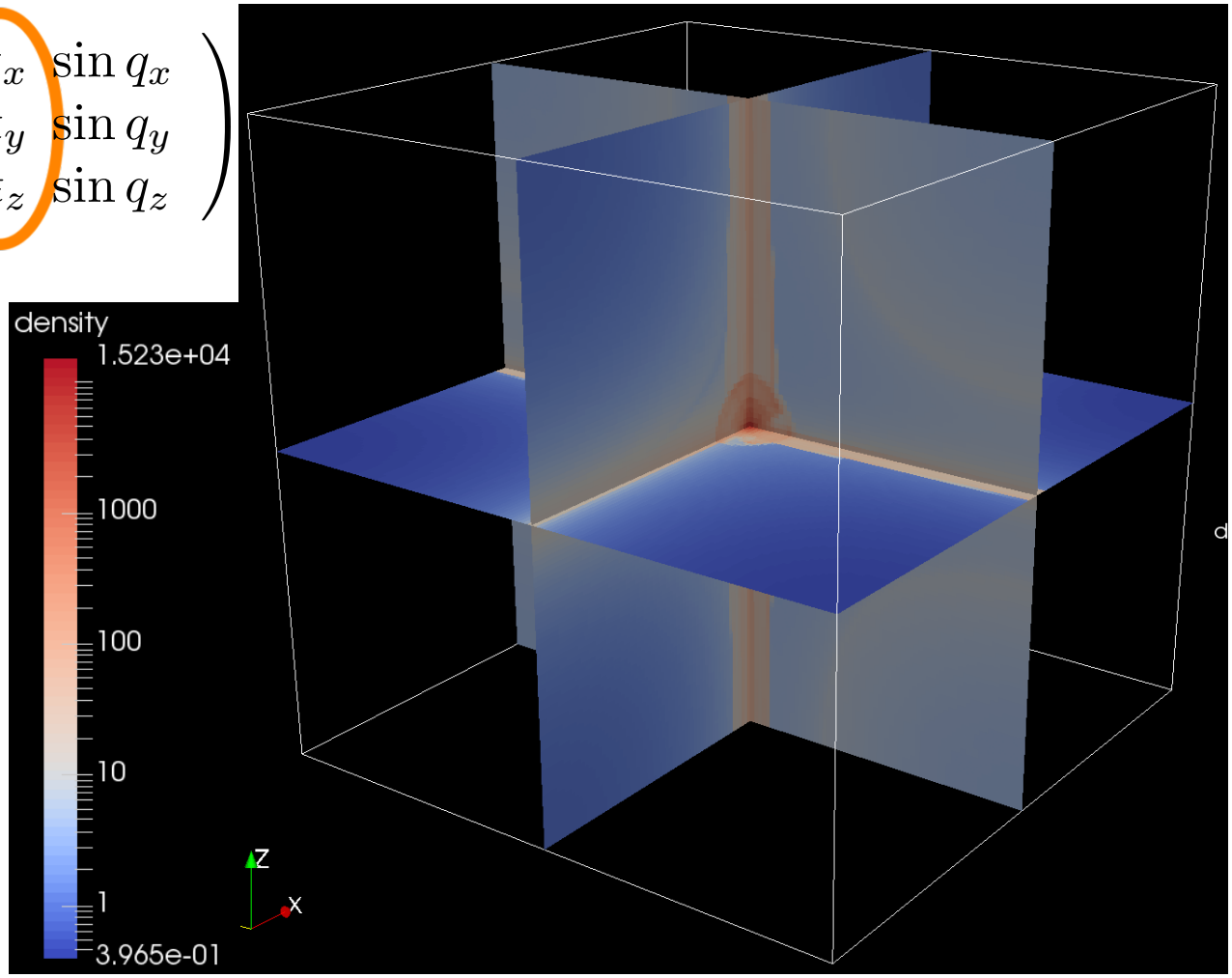


3.1 Initial condition

F.Moutarde et al. [ApJ, 382, 377-381 (1991)]
T.Buchert et al. [A&A, 318, 1-10 (1997)]

Three sine wave:

$$\boldsymbol{\Psi}^{(1)}(\boldsymbol{q},t) = D_+(t) \begin{pmatrix} \epsilon_x \sin q_x \\ \epsilon_y \sin q_y \\ \epsilon_z \sin q_z \end{pmatrix}$$



➔ small single density peak in a periodic boundary box

3.1 Initial condition

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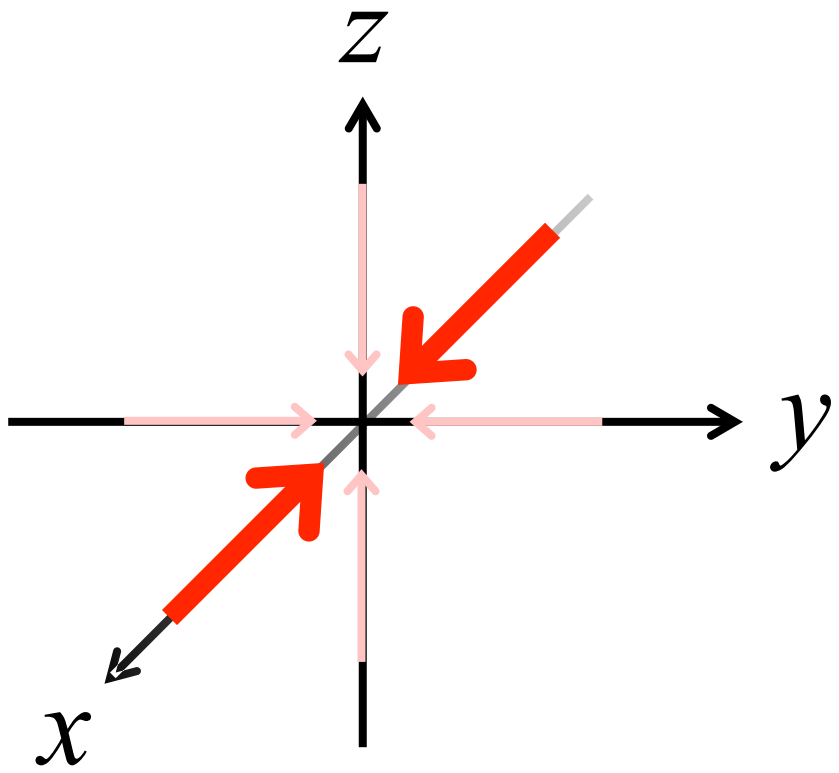
Q1D ($1 \gg \varepsilon_y/\varepsilon_x, \varepsilon_z/\varepsilon_x$)
$(\varepsilon_x, \varepsilon_y, \varepsilon_z) = (-24, -4, -3)$

ASY ($1 \geq \varepsilon_y/\varepsilon_x \geq \varepsilon_z/\varepsilon_x$)
$(\varepsilon_x, \varepsilon_y, \varepsilon_z) = (-24, -18, -12)$

SYM ($1 \sim \varepsilon_y/\varepsilon_x \sim \varepsilon_z/\varepsilon_x$)
$(\varepsilon_x, \varepsilon_y, \varepsilon_z) = (-18, -18, -18)$



Shell-crossing happens along *x*-axis

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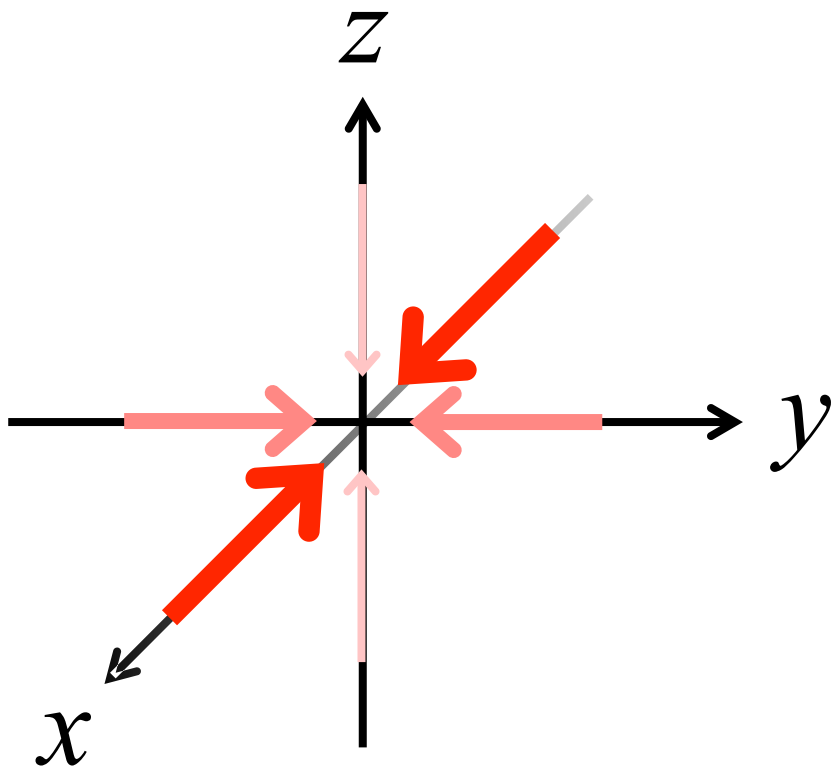

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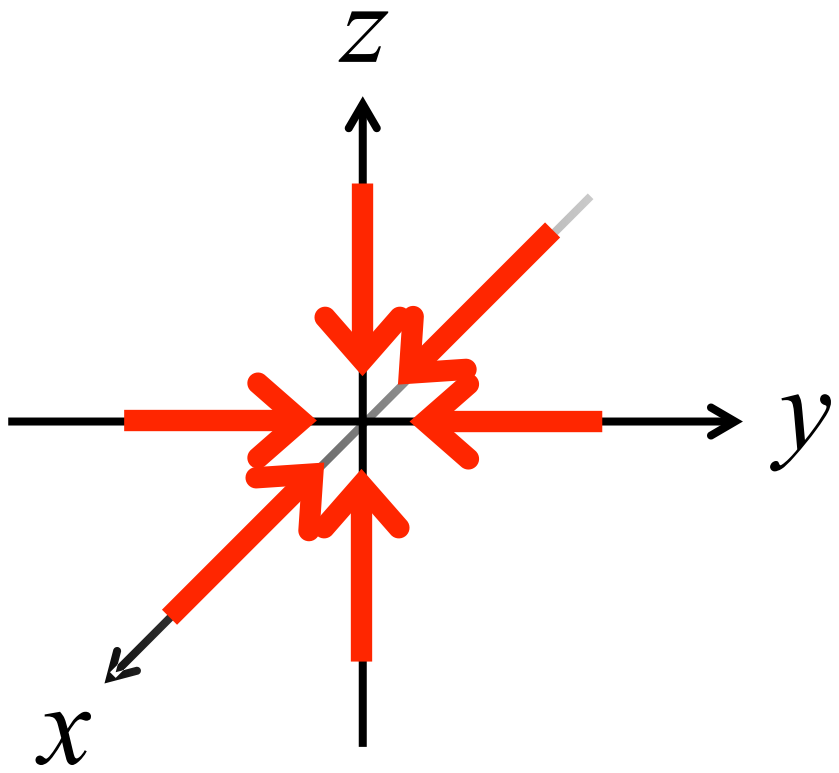

Shell-crossing happens along *x*-, *y*-, and *z*-axes at the same time

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Three sine wave:

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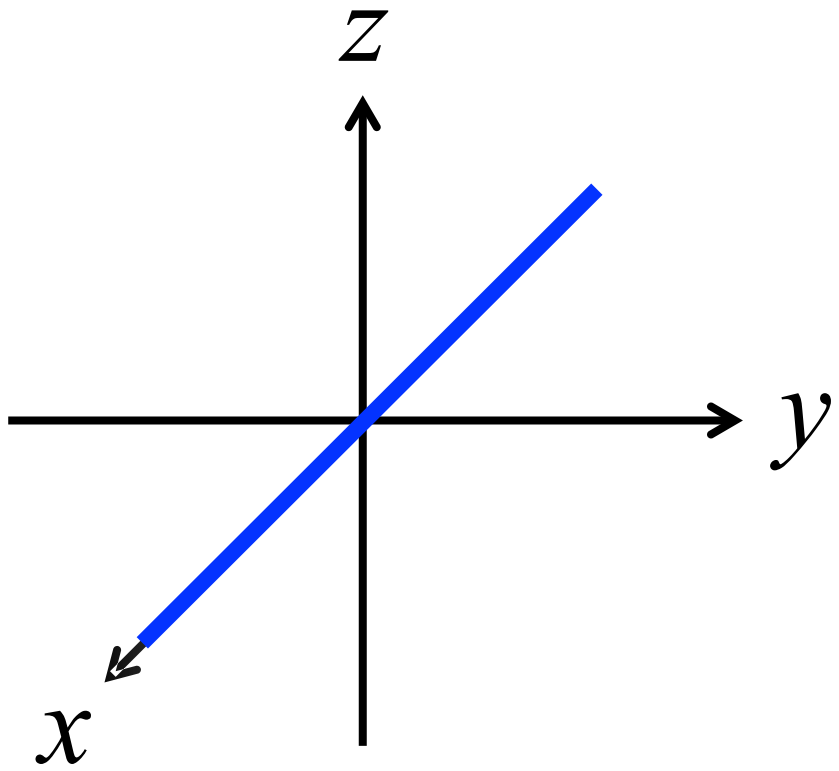
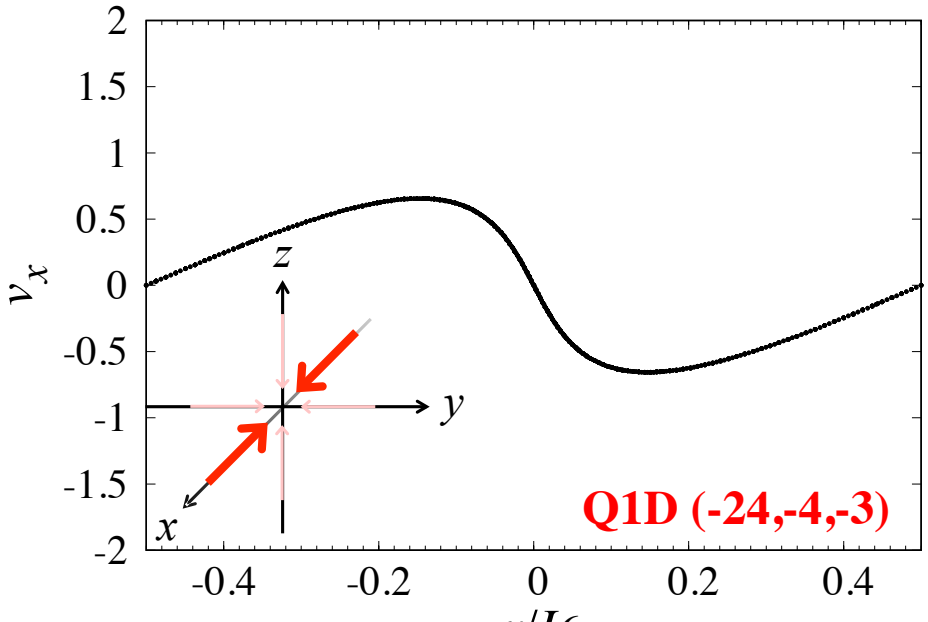
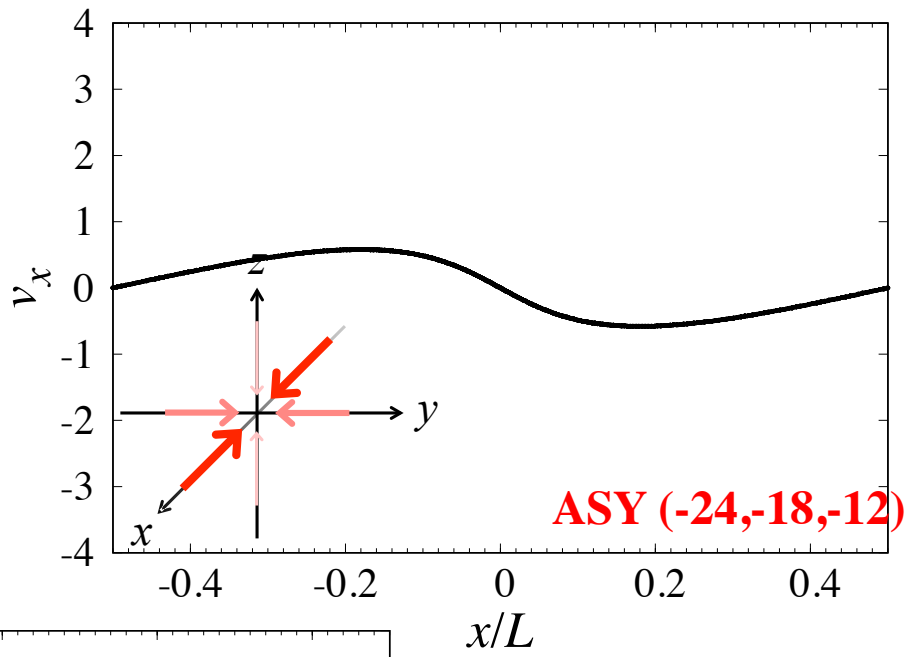
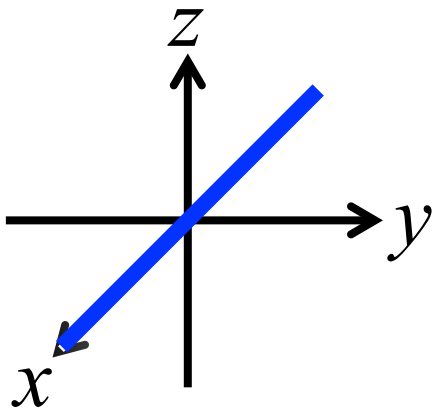


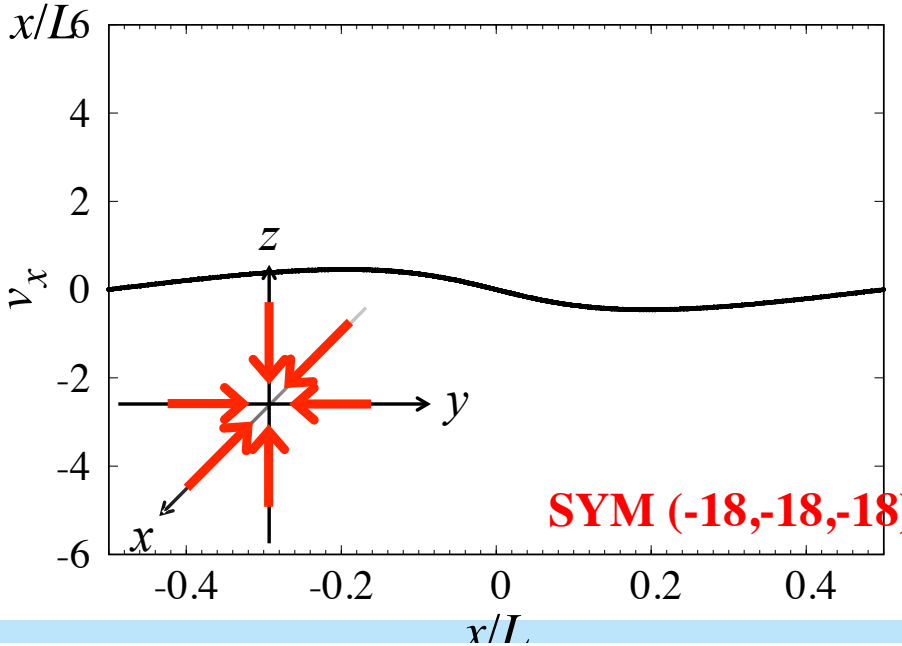
We focus on *x*-v_x subspace of 6D phase-space sheet.

3.2 Vlasov-Poisson simulation (pre-collapse)









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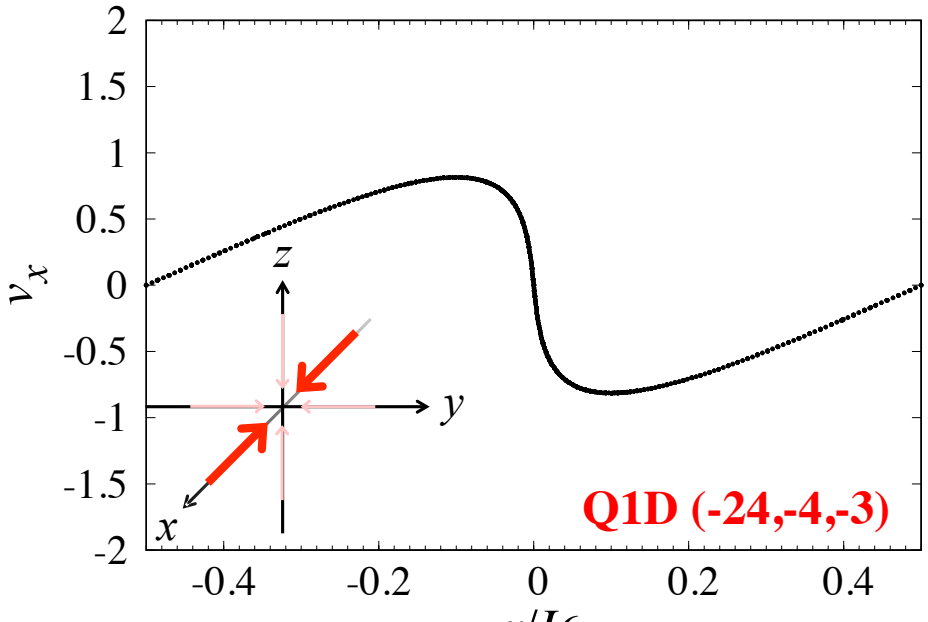
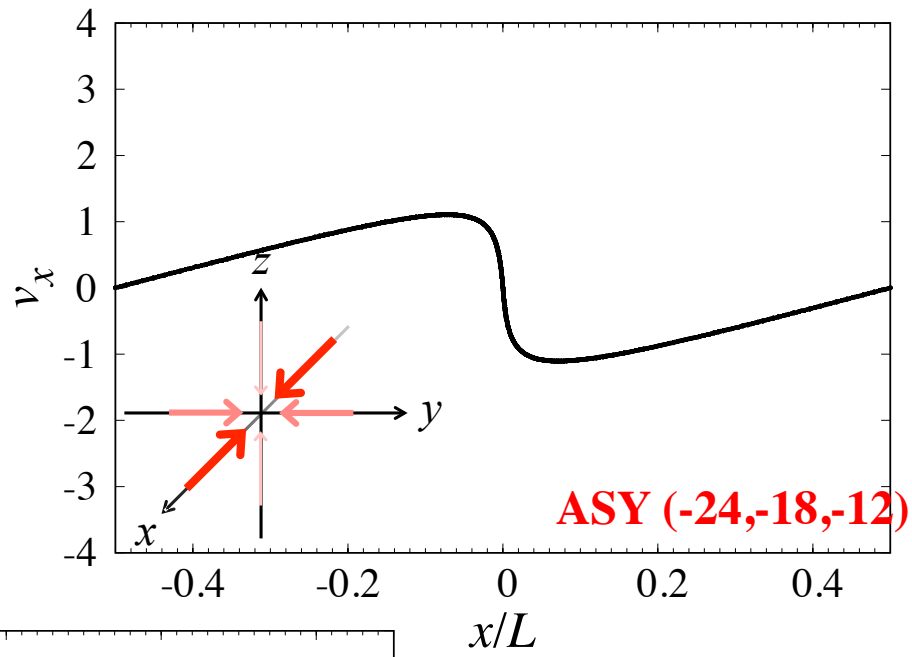
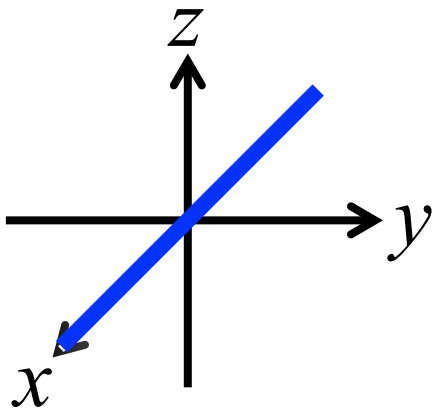
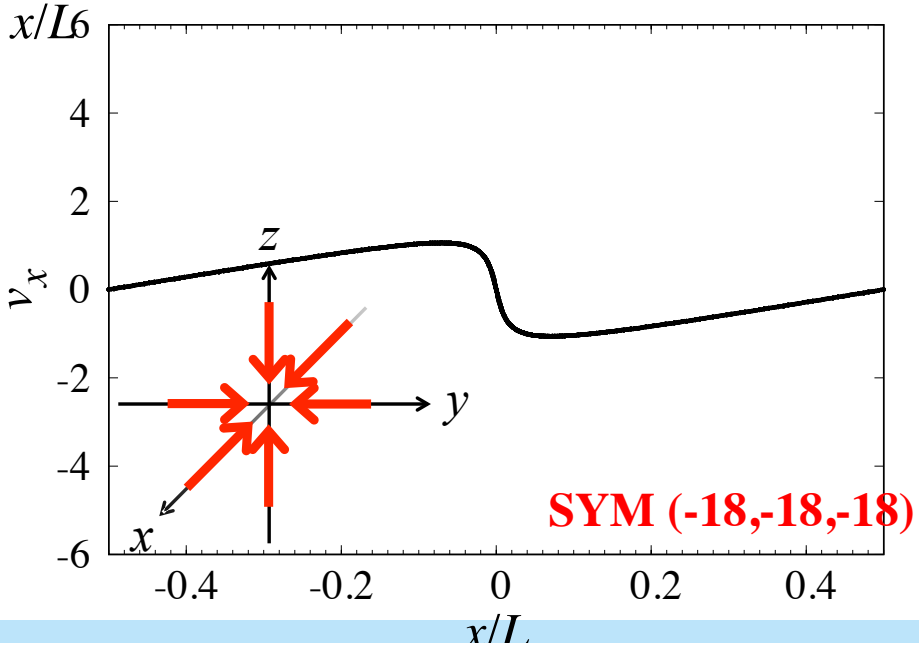
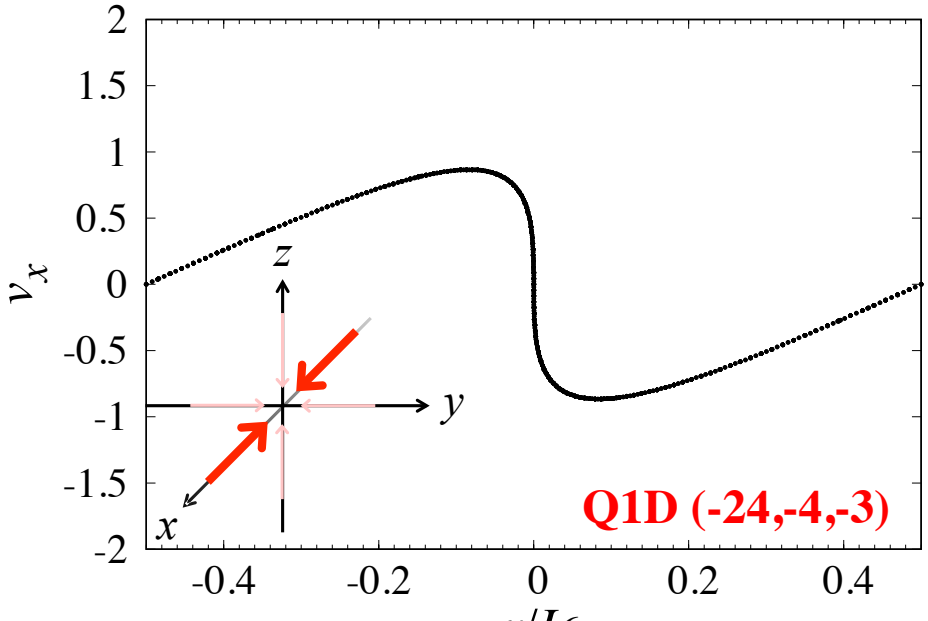


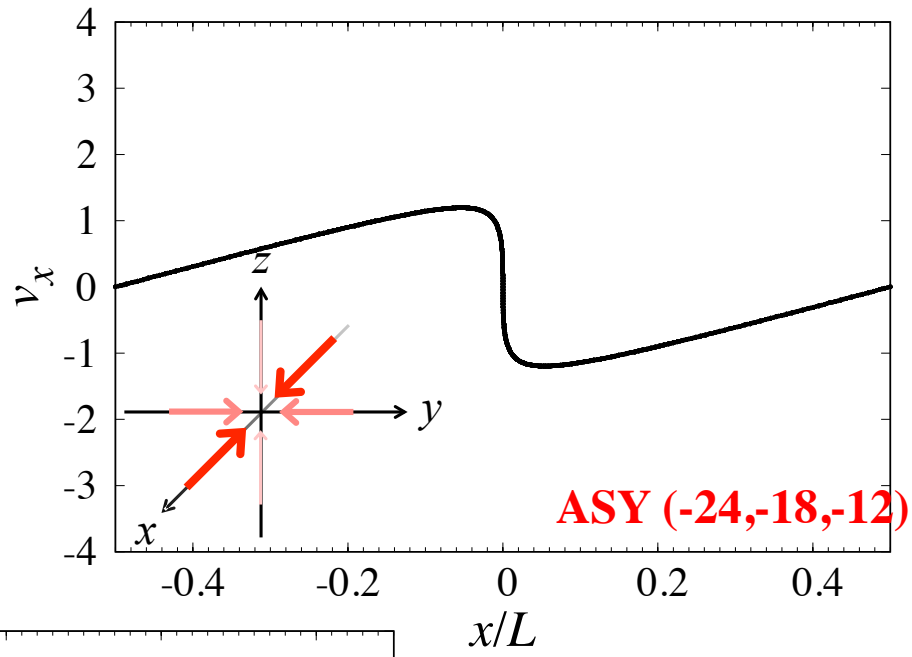


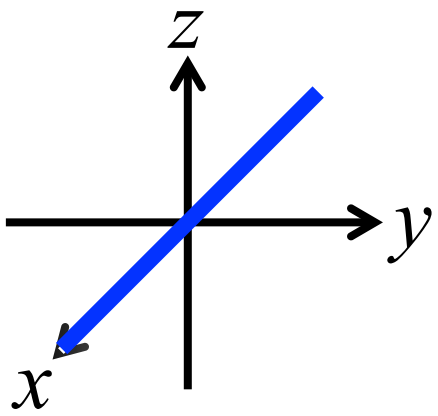


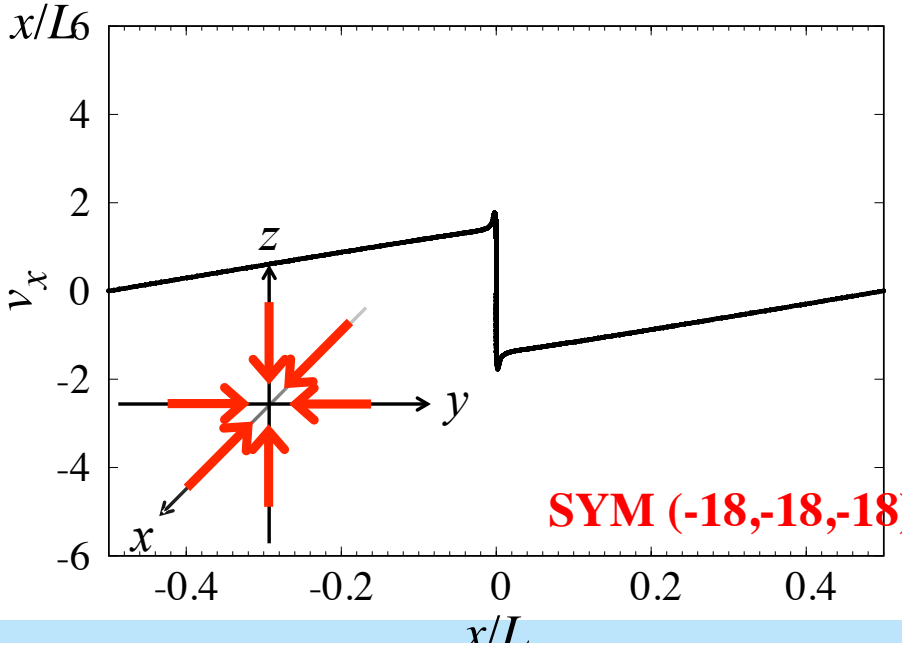


3.2 Vlasov-Poisson simulation (shell-crossing)



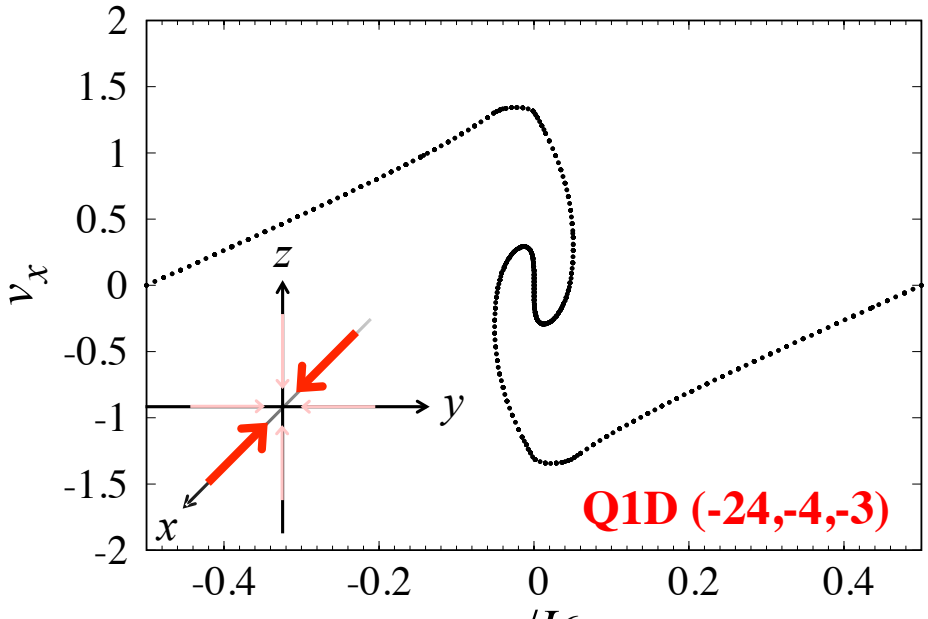


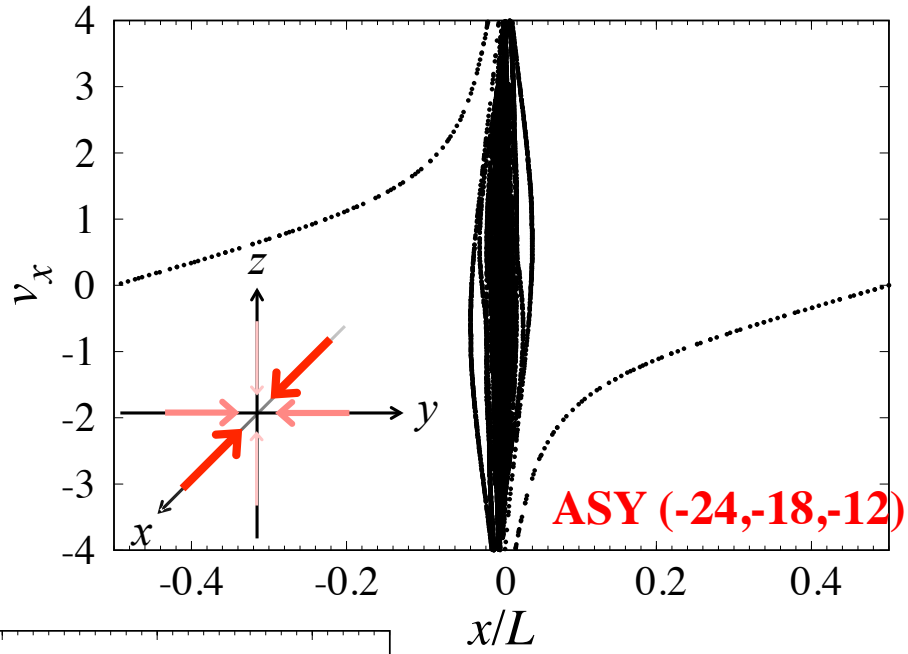


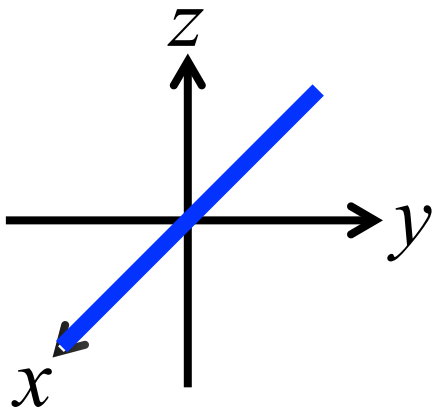


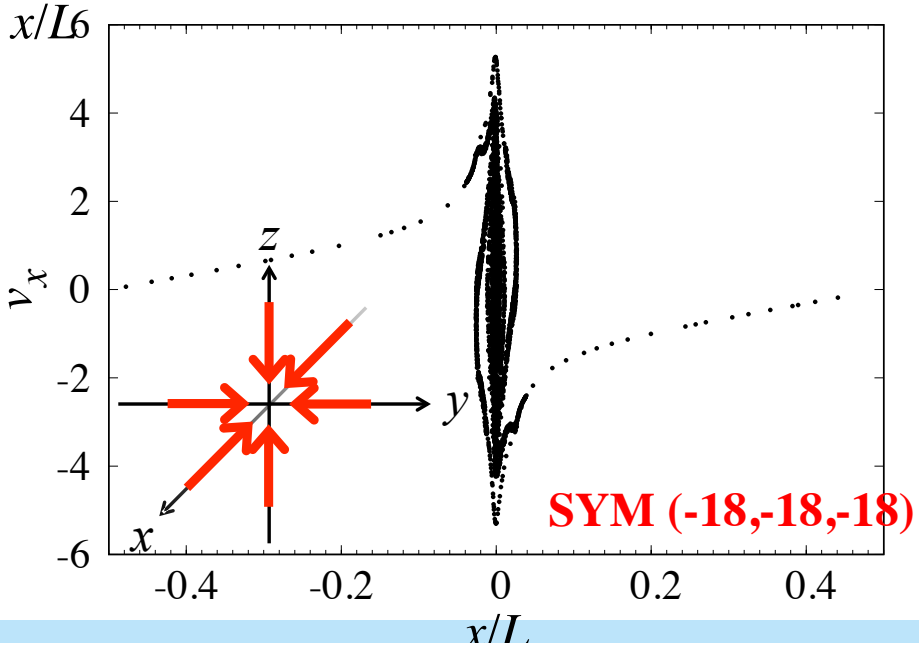
Comment:
Simulation takes about 2days with 3072cores at shell-crossing time for each initial condition.

3.2 Vlasov-Poisson simulation (post-collapse)









4. Results

Three sine wave:

$$\boldsymbol{\Psi}^{(1)}(\boldsymbol{q},t) = D_+(t) \begin{pmatrix} \epsilon_x & \sin q_x \\ \epsilon_y & \sin q_y \\ \epsilon_z & \sin q_z \end{pmatrix}$$

Tractable analytical calculation is possible!

We finally obtain LPT solutions

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up to 10th order.

COMMENT

Our LPT algorithm can be generalized to **any** trigonometric polynomial initial conditions.

4.1 Expressions

x-component of the displacement field

1st order (Zel'dovich approximation)

$$\epsilon_x \sin(q_x)$$

2nd order

$$-\frac{3}{14}\epsilon_x \sin(q_x)\left(\epsilon_y \cos(q_y) + \epsilon_z \cos(q_z)\right)$$

3rd order

$$\frac{\epsilon_x \sin(q_x)\left(78\,\epsilon_x \cos(q_x)\left(\epsilon_y \cos(q_y) + \epsilon_z \cos(q_z)\right) + 160\,\epsilon_y\,\epsilon_z \cos(q_y)\cos(q_z) - 3\,\epsilon_y^2\left(\cos(2\,q_y) - 25\right) - 3\,\epsilon_z^2\left(\cos(2\,q_z) - 25\right)\right)}{1260}$$

4.1 Expressions

4th order

$$-\frac{1}{3\,880\,800}\epsilon_x \sin(q_x)\left(42\,\epsilon_x^2\,(101\cos(2\,q_x)+2123)\left(\epsilon_y\cos(q_y)+\epsilon_z\cos(q_z)\right)+60\,\epsilon_x\cos(q_x)\left(6010\,\epsilon_y\,\epsilon_z\cos(q_y)\cos(q_z)+\epsilon_y^2\left(1274\cos(2\,q_y)+2039\right)+\epsilon_z^2\,(1274\cos(2\,q_z)+2039)\right)+50\,\epsilon_y^2\,\epsilon_z\left(571\cos(2\,q_y)+4177\right)\cos(q_z)+50\,\epsilon_y\,\epsilon_z^2\cos(q_y)\,(571\cos(2\,q_z)+4177)+21\,\epsilon_y^3\left(2715\cos(q_y)-443\cos(3\,q_y)\right)+21\,\epsilon_z^3\,(2715\cos(q_z)-443\cos(3\,q_z))\right)$$

5th order

$$\frac{1}{18\,396\,738\,360\,000}\epsilon_x\sin(q_x)\,(54\,054\,\epsilon_x^3\,(1\,970\,751\cos(q_x)-126\,335\cos(3\,q_x))\left(\epsilon_y\cos(q_y)+\epsilon_z\cos(q_z)\right)+765\,\epsilon_x^2\left(1040\,\epsilon_y\,\epsilon_z\,(203\,958\cos(2\,q_x)+1\,102\,715)\cos(q_y)\cos(q_z)+33\,\epsilon_y^2\left(2\,273\,783\cos\left(2\left(q_x-q_y\right)\right)+2\,273\,783\cos\left(2\left(q_x+q_y\right)\right)+1\,476\,930\cos(2\,q_x)+9\,562\,610\cos(2\,q_y)+9\,599\,590\right)+33\,\epsilon_z^2\,(2\,273\,783\cos(2\,(q_x-q_z))+2\,273\,783\cos(2\,(q_x+q_z))+1\,476\,930\cos(2\,q_x)+9\,562\,610\cos(2\,q_z)+9\,599\,590)\right)+1870\,\epsilon_x\cos(q_x)\left(26\,\epsilon_y^2\,\epsilon_z\left(11\,522\,545\cos(2\,q_y)+20\,694\,771\right)\cos(q_z)+26\,\epsilon_y\,\epsilon_z^2\cos(q_y)\,(11\,522\,545\cos(2\,q_z)+20\,694\,771)+27\,\epsilon_y^3\left(11\,773\,099\cos(q_y)-369\,227\cos(3\,q_y)\right)+27\,\epsilon_z^3\,(11\,773\,099\cos(q_z)-369\,227\cos(3\,q_z))\right)-13\left(214\,200\,\epsilon_y^3\,\epsilon_z\left(19\,521\cos(3\,q_y)-180\,697\cos(q_y)\right)\cos(q_z)-1870\,\epsilon_y^2\,\epsilon_z^2\left(1\,919\,335\cos\left(2\left(q_y-q_z\right)\right)+1\,919\,335\cos\left(2\left(q_y+q_z\right)\right)+4\,759\,758\cos\left(2\,q_y\right)+4\,759\,758\cos(2\,q_z)+21\,165\,750\right)+214\,200\,\epsilon_y\,\epsilon_z^3\cos(q_y)\,(19\,521\cos(3\,q_z)-180\,697\cos(q_z))+2079\,\epsilon_y^4\left(1\,297\,372\cos\left(2\,q_y\right)+59\,485\cos\left(4\,q_y\right)-3\,461\,625\right)+2079\,\epsilon_z^4\,(1\,297\,372\cos(2\,q_z)+59\,485\cos(4\,q_z)-3\,461\,625)))$$

4.1 Expressions

6th order

$$
\frac{1}{811\,296\,161\,676\,000\,000}
\sin(q_x)\,\epsilon_x\,(-305\,613\,(2\,659\,396\,870\cos(q_y) - 962\,623\,441\cos(3\,q_y) + 28\,312\,075\cos(5\,q_y))\,\epsilon_y^5 + 4\,873\,050\,(266\,598\,492\cos(2\,q_y) + 76\,018\,045\cos(4\,q_y) - 1\,901\,498\,841)\cos(q_z)\,\epsilon_z\,\epsilon_y^4 -
4420\cos(q_y)\,(-516\,334\,675\,626\cos(2\,q_y) + 52\,110\,906\,915\cos(2\,(q_y - q_z)) + 1\,838\,044\,474\,430\cos(2\,q_z) + 52\,110\,906\,915\cos(2\,(q_y + q_z)) + 3\,604\,788\,111\,078)\,\epsilon_z^2\,\epsilon_y^3 -
4420\cos(q_z)\,(1\,838\,044\,474\,430\cos(2\,q_y) + 52\,110\,906\,915\cos(2\,(q_y - q_z)) - 516\,334\,675\,626\cos(2\,q_z) + 52\,110\,906\,915\cos(2\,(q_y + q_z)) + 3\,604\,788\,111\,078)\,\epsilon_z^3\,\epsilon_y^2 +
4\,873\,050\cos(q_y)\,(266\,598\,492\cos(2\,q_z) + 76\,018\,045\cos(4\,q_z) - 1\,901\,498\,841)\,\epsilon_z^4\,\epsilon_y - 305\,613\,(2\,659\,396\,870\cos(q_z) - 962\,623\,441\cos(3\,q_z) + 28\,312\,075\cos(5\,q_z))\,\epsilon_z^5 +
1\,833\,678\,(116\,189\,716\cos(2\,q_x) + 7\,558\,375\cos(4\,q_x) - 839\,506\,587)\,\epsilon_x^4\,(\cos(q_y)\,\epsilon_y + \cos(q_z)\,\epsilon_z) + 8820\cos(q_x)\,\epsilon_x^3
(-33\,(-1\,499\,341\,220\cos(2\,q_x) + 745\,268\,925\cos(2\,(q_x - q_y)) + 34\,417\,340\,854\cos(2\,q_y) + 745\,268\,925\cos(2\,(q_x + q_y)) + 15\,828\,123\,428)\,\epsilon_y^2 + 4420\,(32\,910\,965\cos(2\,q_x) - 531\,579\,429)
\cos(q_y)\cos(q_z)\,\epsilon_z\,\epsilon_y - 33\,(-1\,499\,341\,220\cos(2\,q_x) + 745\,268\,925\cos(2\,(q_x - q_z)) + 34\,417\,340\,854\cos(2\,q_z) + 745\,268\,925\cos(2\,(q_x + q_z)) + 15\,828\,123\,428)\,\epsilon_z^2) -
340\,\epsilon_x^2\,(2079\cos(q_y)\,(8\,156\,203\,254\cos(2\,q_x) + 2\,768\,321\,325\cos(2\,(q_x - q_y)) + 2\,763\,034\,018\cos(2\,q_y) + 2\,768\,321\,325\cos(2\,(q_x + q_y)) + 21\,975\,421\,138)\,\epsilon_y^3 +
52\,(386\,519\,295\,393\cos(2\,q_x) + 295\,586\,482\,530\cos(2\,(q_x - q_y)) + 1\,129\,466\,363\,210\cos(2\,q_y) + 295\,586\,482\,530\cos(2\,(q_x + q_y)) + 1\,479\,850\,604\,694)\cos(q_z)\,\epsilon_z\,\epsilon_y^2 +
52\cos(q_y)\,(386\,519\,295\,393\cos(2\,q_x) + 295\,586\,482\,530\cos(2\,(q_x - q_z)) + 1\,129\,466\,363\,210\cos(2\,q_z) + 295\,586\,482\,530\cos(2\,(q_x + q_z)) + 1\,479\,850\,604\,694)\,\epsilon_z^2\,\epsilon_y +
2079\cos(q_z)\,(8\,156\,203\,254\cos(2\,q_x) + 2\,768\,321\,325\cos(2\,(q_x - q_z)) + 2\,763\,034\,018\cos(2\,q_z) + 2\,768\,321\,325\cos(2\,(q_x + q_z)) + 21\,975\,421\,138)\,\epsilon_z^3) -
20\cos(q_x)\,\epsilon_x\,(14\,553\,(5\,904\,767\,794\cos(2\,q_y) - 2\,780\,948\,475\cos(4\,q_y) + 39\,272\,806\,105)\,\epsilon_y^4 + 4420\,(571\,117\,785\,469\cos(q_y) - 1\,357\,574\,367\cos(3\,q_y))\cos(q_z)\,\epsilon_z\,\epsilon_y^3 +
510\,510\,(1\,660\,817\,391\cos(2\,q_y) + 538\,260\,884\cos(2\,(q_y - q_z)) + 1\,660\,817\,391\cos(2\,q_z) + 538\,260\,884\cos(2\,(q_y + q_z)) + 2\,964\,557\,714)\,\epsilon_z^2\,\epsilon_y^2 +
4420\cos(q_y)\,(571\,117\,785\,469\cos(q_z) - 1\,357\,574\,367\cos(3\,q_z))\,\epsilon_z^3\,\epsilon_y + 14\,553\,(5\,904\,767\,794\cos(2\,q_z) - 2\,780\,948\,475\cos(4\,q_z) + 39\,272\,806\,105)\,\epsilon_z^4))
$$

4.1 Expressions

7th order

$$
\frac{1}{1\,004\,810\,024\,548\,828\,360\,142\,280\,000\,000}\sin(q_x)\,\epsilon_x\,(3\,034\,100\,803\,734\,(132\,504\,410\,196\,934\cos(q_x)-22\,415\,880\,538\,705\cos(3\,q_x)+383\,378\,540\,875\cos(5\,q_x))\,(\cos(q_y)\,\epsilon_y+\cos(q_z)\,\epsilon_z)\,\epsilon_x^5+
$$

$$
553\,391\,055\,(363\,(277\,129\,358\,703\,288\cos(2\,q_x)-221\,569\,332\,279\,750\cos(4\,q_x)-310\,443\,780\,648\,625\cos(4\,q_x-2\,q_y)+4\,043\,457\,776\,606\,492\cos(2\,(q_x-q_y))+26\,302\,111\,171\,118\,442\cos(2\,q_y)+
$$

$$
4\,043\,457\,776\,606\,492\cos(2\,(q_x+q_y))-310\,443\,780\,648\,625\cos(2\,(2\,q_x+q_y))+8\,639\,970\,224\,710\,734)\,\epsilon_y^2+392\,080\,(2\,849\,830\,451\,562\cos(2\,q_x)-818\,223\,510\,775\cos(4\,q_x)+44\,804\,309\,653\,645)
$$

$$
\cos(q_y)\cos(q_z)\,\epsilon_z\,\epsilon_y+363\,(277\,129\,358\,703\,288\cos(2\,q_x)-221\,569\,332\,279\,750\cos(4\,q_x)-310\,443\,780\,648\,625\cos(4\,q_x-2\,q_z)+4\,043\,457\,776\,606\,492\cos(2\,(q_x-q_z))+
$$

$$
26\,302\,111\,171\,118\,442\cos(2\,q_z)+4\,043\,457\,776\,606\,492\cos(2\,(q_x+q_z))-310\,443\,780\,648\,625\cos(2\,(2\,q_x+q_z))+8\,639\,970\,224\,710\,734)\,\epsilon_z^2)\,\epsilon_x^4+
$$

$$
4\,485\,140\cos(q_x)\,(87\,318\cos(q_y)\,(17\,(317\,200\,143\,950\,625\cos(2\,(q_x-q_y))+1\,423\,122\,370\,938\,650\cos(2\,q_y)+317\,200\,143\,950\,625\cos(2\,(q_x+q_y))+2\,162\,942\,532\,366\,438)-2\,323\,267\,677\,375\,190\cos(2\,q_x))\,\epsilon_y^3+
$$

$$
130\,(17\,(137\,983\,685\,717\,505\,525\cos(2\,(q_x-q_y))+2\,969\,669\,587\,394\,985\,734\cos(2\,q_y)+137\,983\,685\,717\,505\,525\cos(2\,(q_x+q_y))+2\,215\,505\,888\,579\,262\,006)-1\,048\,826\,181\,631\,251\,114\cos(2\,q_x))
$$

$$
\cos(q_z)\,\epsilon_z\,\epsilon_y^2+130\cos(q_y)
$$

$$
(17\,(137\,983\,685\,717\,505\,525\cos(2\,(q_x-q_z))+2\,969\,669\,587\,394\,985\,734\cos(2\,q_z)+137\,983\,685\,717\,505\,525\cos(2\,(q_x+q_z))+2\,215\,505\,888\,579\,262\,006)-1\,048\,826\,181\,631\,251\,114\cos(2\,q_x))\,\epsilon_z^2\,\epsilon_y+
$$

$$
87\,318\cos(q_z)\,(17\,(317\,200\,143\,950\,625\cos(2\,(q_x-q_z))+1\,423\,122\,370\,938\,650\cos(2\,q_z)+317\,200\,143\,950\,625\cos(2\,(q_x+q_z))+2\,162\,942\,532\,366\,438)-2\,323\,267\,677\,375\,190\cos(2\,q_x))\,\epsilon_z^3)\,\epsilon_x^3+
$$

$$
75\,110\,(1\,303\,533\,(58\,152\,516\,016\,142\,730\cos(2\,q_x)-1\,031\,493\,519\,759\,625\cos(2\,(q_x-2\,q_y))+28\,800\,768\,510\,344\,900\cos(2\,(q_x-q_y))+48\,549\,817\,895\,634\,696\cos(2\,q_y)-
$$

$$
7\,124\,418\,190\,183\,710\cos(4\,q_y)+28\,800\,768\,510\,344\,900\cos(2\,(q_x+q_y))-1\,031\,493\,519\,759\,625\cos(2\,(q_x+2\,q_y))+104\,493\,594\,962\,874\,390)\,\epsilon_y^4+1\,127\,100\cos(q_y)
$$

$$
(226\,760\,047\,458\,931\,834\cos(2\,q_x)+63\,226\,570\,697\,650\,635\cos(2\,(q_x-q_y))+85\,203\,185\,368\,922\,438\cos(2\,q_y)+63\,226\,570\,697\,650\,635\cos(2\,(q_x+q_y))+536\,755\,303\,105\,443\,074)\cos(q_z)\,\epsilon_z\,\epsilon_y^3+
$$

$$
461\,890\,(241\,590\,221\,276\,456\,052\cos(2\,q_x)+145\,469\,210\,965\,983\,954\cos(2\,(q_x-q_y))+547\,592\,158\,923\,252\,492\cos(2\,q_y)+145\,469\,210\,965\,983\,954\cos(2\,(q_x+q_y))+
$$

$$
145\,469\,210\,965\,983\,954\cos(2\,(q_x-q_z))+61\,752\,265\,120\,742\,625\cos(2\,(q_x-q_y-q_z))+190\,476\,735\,598\,070\,350\cos(2\,(q_y-q_z))+
$$

$$
61\,752\,265\,120\,742\,625\cos(2\,(q_x+q_y-q_z))+547\,592\,158\,923\,252\,492\cos(2\,q_z)+145\,469\,210\,965\,983\,954\cos(2\,(q_x+q_z))+61\,752\,265\,120\,742\,625\cos(2\,(q_x-q_y+q_z))+
$$

$$
190\,476\,735\,598\,070\,350\cos(2\,(q_y+q_z))+61\,752\,265\,120\,742\,625\cos(2\,(q_x+q_y+q_z))+722\,801\,721\,130\,204\,716)\,\epsilon_z^2\,\epsilon_y^2+1\,127\,100\cos(q_y)\cos(q_z)
$$

$$
(226\,760\,047\,458\,931\,834\cos(2\,q_x)+63\,226\,570\,697\,650\,635\cos(2\,(q_x-q_z))+85\,203\,185\,368\,922\,438\cos(2\,q_z)+63\,226\,570\,697\,650\,635\cos(2\,(q_x+q_z))+536\,755\,303\,105\,443\,074)\,\epsilon_z^3\,\epsilon_y+
$$

$$
1\,303\,533\,(58\,152\,516\,016\,142\,730\cos(2\,q_x)-1\,031\,493\,519\,759\,625\cos(2\,(q_x-2\,q_z))+28\,800\,768\,510\,344\,900\cos(2\,(q_x-q_z))+48\,549\,817\,895\,634\,696\cos(2\,q_z)-
$$

$$
7\,124\,418\,190\,183\,710\cos(4\,q_z)+28\,800\,768\,510\,344\,900\cos(2\,(q_x+q_z))-1\,031\,493\,519\,759\,625\cos(2\,(q_x+2\,q_z))+104\,493\,594\,962\,874\,390)\,\epsilon_z^4)\,\epsilon_x^2+
$$

$$
1\,314\,610\cos(q_x)\,(-305\,613\,(-22\,055\,935\,460\,016\,250\cos(q_y)+5\,283\,563\,479\,587\,967\cos(3\,q_y)+218\,655\,867\,954\,875\cos(5\,q_y))\,\epsilon_y^5+
$$

$$
3770\,(1\,730\,167\,322\,158\,143\,988\cos(2\,q_y)-429\,044\,746\,416\,822\,975\cos(4\,q_y)+6\,421\,215\,185\,231\,777\,931)\cos(q_z)\,\epsilon_z\,\epsilon_y^4+
$$

$$
52\,780\cos(q_y)\,(24\,468\,715\,615\,587\,642\cos(2\,q_y)+5\,(6\,222\,005\,410\,021\,245\cos(2\,(q_y-q_z))+82\,009\,905\,587\,163\,710\cos(2\,q_z)+6\,222\,005\,410\,021\,245\cos(2\,(q_y+q_z))+135\,736\,584\,119\,121\,846))\,\epsilon_z^2\,\epsilon_y^3+
$$

$$
52\,780\cos(q_z)\,(410\,049\,527\,935\,818\,550\cos(2\,q_y)+31\,110\,027\,050\,106\,225\cos(2\,(q_y-q_z))+24\,468\,715\,615\,587\,642\cos(2\,q_z)+31\,110\,027\,050\,106\,225\cos(2\,(q_y+q_z))+678\,682\,920\,595\,609\,230)\,\epsilon_z^3\,\epsilon_y^2+
$$

$$
3770\cos(q_y)\,(1\,730\,167\,322\,158\,143\,988\cos(2\,q_z)-429\,044\,746\,416\,822\,975\cos(4\,q_z)+6\,421\,215\,185\,231\,777\,931)\,\epsilon_z^4\,\epsilon_y-
$$

$$
305\,613\,(-22\,055\,935\,460\,016\,250\cos(q_z)+5\,283\,563\,479\,587\,967\cos(3\,q_z)+218\,655\,867\,954\,875\cos(5\,q_z))\,\epsilon_z^5)\,\epsilon_x+
$$

$$
377\,(1\,341\,335\,457\,(-443\,913\,672\,901\,179\cos(2\,q_y)+6\,195\,867\,562\,590\cos(4\,q_y)+425\,(2\,828\,142\,515\cos(6\,q_y)+1\,790\,837\,852\,258))\,\epsilon_y^6+
$$

$$
143\,881\,674\,300\,(83\,805\,177\,792\,886\cos(q_y)-24\,017\,169\,020\,137\cos(3\,q_y)+105\,584\,014\,195\cos(5\,q_y))\cos(q_z)\,\epsilon_z\,\epsilon_y^5-
$$

$$
193\,325\,(3\,108\,729\,593\,158\,423\,416\cos(2\,q_y)+5\,743\,038\,318\,082\,194\,402\cos(4\,q_y)+17\,(134\,073\,949\,961\,514\,975\cos(4\,q_y-2\,q_z)-825\,388\,193\,136\,040\,604\cos(2\,(q_y-q_z))-
$$

$$
4\,531\,039\,763\,474\,630\,406\cos(2\,q_z)-825\,388\,193\,136\,040\,604\cos(2\,(q_y+q_z))+134\,073\,949\,961\,514\,975\cos(2\,(2\,q_y+q_z))-6\,805\,749\,817\,327\,430\,970))\,\epsilon_z^2\,\epsilon_y^4+449\,106\,000\cos(q_y)\cos(q_z)
$$

$$
(-2\,893\,688\,202\,539\,726\cos(2\,q_y)+7\,989\,174\,900\,132\,525\cos(2\,(q_y-q_z))-2\,893\,688\,202\,539\,726\cos(2\,q_z)+7\,989\,174\,900\,132\,525\cos(2\,(q_y+q_z))+130\,989\,210\,802\,201\,794)\,\epsilon_z^3\,\epsilon_y^3+
$$

$$
193\,325\,(77\,027\,675\,979\,068\,716\,902\cos(2\,q_y)-2\,279\,257\,149\,345\,754\,575\cos(2\,(q_y-2\,q_z))+14\,031\,599\,283\,312\,690\,268\cos(2\,(q_y-q_z))-3\,108\,729\,593\,158\,423\,416\cos(2\,q_z)-
$$

$$
5\,743\,038\,318\,082\,194\,402\cos(4\,q_z)+14\,031\,599\,283\,312\,690\,268\cos(2\,(q_y+q_z))-2\,279\,257\,149\,345\,754\,575\cos(2\,(q_y+2\,q_z))+115\,697\,746\,894\,566\,326\,490)\,\epsilon_z^4\,\epsilon_y^2+
$$

$$
143\,881\,674\,300\cos(q_y)\,(83\,805\,177\,792\,886\cos(q_z)-24\,017\,169\,020\,137\cos(3\,q_z)+105\,584\,014\,195\cos(5\,q_z))\,\epsilon_z^5\,\epsilon_y+
$$

$$
1\,341\,335\,457\,(-443\,913\,672\,901\,179\cos(2\,q_z)+6\,195\,867\,562\,590\cos(4\,q_z)+425\,(2\,828\,142\,515\cos(6\,q_z)+1\,790\,837\,852\,258))\,\epsilon_z^6))
$$

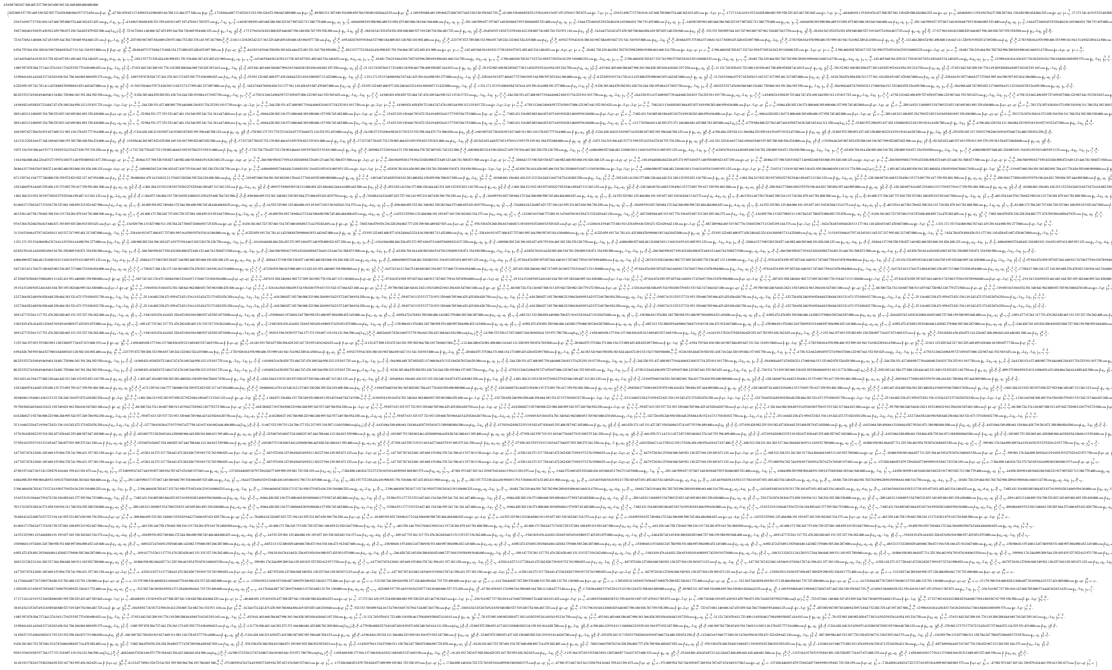
4.1 Expressions

8th order

$$
\begin{aligned}
&\frac{1}{10\,691\,178\,661\,199\,533\,751\,913\,859\,200\,000\,000}\\
&\sin(q_x)\,\epsilon_x\,(1\,633\,746\,586\,626\,(-245\,713\,485\,247\,929\,951\cos(2q_x)-2\,195\,281\,540\,970\,250\cos(4q_x)+494\,330\,346\,828\,175\cos(6q_x)+865\,027\,121\,950\,584\,922)\,(\cos(q_y)\,\epsilon_y+\cos(q_z)\,\epsilon_z)\,\epsilon_x^6-7\,779\,240\cos(q_x)\,(627\,(428\,992\,591\,700\,823\,991\,100\cos(2q_x)+1\,571\,794\,373\,829\,512\,625\cos(4q_x)+\\
&\qquad 33\,352\,805\,228\,223\,745\,550\cos(4q_x-2q_y)+515\,931\,466\,598\,138\,198\,040\cos(2\,(q_x-q_y))-7\,784\,580\,247\,782\,371\,085\,868\cos(2q_y)+515\,931\,466\,598\,138\,198\,040\cos(2\,(q_x+q_y))+33\,352\,805\,228\,223\,745\,550\cos(2\,(2q_x+q_y))-1\,529\,203\,372\,519\,416\,608\,365)\,\epsilon_y^2+\\
&\qquad 2\,371\,330\,(560\,926\,390\,417\,289\,924\cos(2q_x)+2\,752\,127\,183\,932\,775\cos(4q_x)-2\,607\,108\,359\,711\,238\,507)\cos(q_y)\cos(q_z)\,\epsilon_z\,\epsilon_y+627\,(428\,992\,591\,700\,823\,991\,100\cos(2q_x)+1\,571\,794\,373\,829\,512\,625\cos(4q_x)+33\,352\,805\,228\,223\,745\,550\cos(4q_x-2q_z)+\\
&\qquad 515\,931\,466\,598\,138\,198\,040\cos(2\,(q_x-q_z))-7\,784\,580\,247\,782\,371\,085\,868\cos(2q_z)+515\,931\,466\,598\,138\,198\,040\cos(2\,(q_x+q_z))+33\,352\,805\,228\,223\,745\,550\cos(2\,(2q_x+q_z))-1\,529\,203\,372\,519\,416\,608\,365)\,\epsilon_z^2)\,\epsilon_x^5+\\
&\qquad 3515\,(4\,584\,195\,(1\,037\,796\,098\,184\,111\,135\,612\cos(2q_x-3q_y)+22\,425\,047\,626\,060\,301\,625\cos(4q_x-3q_y)+1\,065\,982\,814\,286\,662\,929\,188\cos(2q_x-q_y)-71\,045\,518\,667\,401\,407\,385\cos(4q_x-q_y)+5\,558\,973\,502\,654\,370\,204\,522\cos(q_y)+\\
&\qquad 2\,436\,591\,489\,369\,587\,854\,102\cos(3q_y)+1\,065\,982\,814\,286\,662\,929\,188\cos(2q_x+q_y)-71\,045\,518\,667\,401\,407\,385\cos(4q_x+q_y)+1\,037\,796\,098\,184\,111\,135\,612\cos(2q_x+3q_y)+22\,425\,047\,626\,060\,301\,625\cos(4q_x+3q_y))\,\epsilon_y^3+\\
&\qquad 58\,(85\,725\,090\,737\,450\,548\,645\,079\,448\cos(2q_x)-11\,788\,106\,125\,102\,093\,582\,384\,050\cos(4q_x)-7\,409\,385\,947\,584\,965\,562\,593\,375\cos(4q_x-2q_y)+169\,968\,401\,151\,646\,360\,846\,947\,780\cos(2\,(q_x-q_y))+805\,731\,257\,316\,020\,840\,390\,800\,630\cos(2q_y)+169\,968\,401\,151\,646\,360\,846\,947\,780\cos(2\,(q_x+q_y))-\\
&\qquad 7\,409\,385\,947\,584\,965\,562\,593\,375\cos(2\,(2q_x+q_y))+471\,288\,301\,537\,253\,467\,508\,400\,474)\cos(q_z)\,\epsilon_z\,\epsilon_y^2+58\cos(q_y)\,(85\,725\,090\,737\,450\,548\,645\,079\,448\cos(2q_x)-11\,788\,106\,125\,102\,093\,582\,384\,050\cos(4q_x)-7\,409\,385\,947\,584\,965\,562\,593\,375\cos(4q_x-2q_z)+\\
&\qquad 169\,968\,401\,151\,646\,360\,846\,947\,780\cos(2\,(q_x-q_z))+805\,731\,257\,316\,020\,840\,390\,800\,630\cos(2q_z)+169\,968\,401\,151\,646\,360\,846\,947\,780\cos(2\,(q_x+q_z))-7\,409\,385\,947\,584\,965\,562\,593\,375\cos(2\,(2q_x+q_z))+471\,288\,301\,537\,253\,467\,508\,400\,474)\,\epsilon_z^2\,\epsilon_y+\\
&\qquad 4\,584\,195\,(1\,037\,796\,098\,184\,111\,135\,612\cos(2q_x-3q_z)+22\,425\,047\,626\,060\,301\,625\cos(4q_x-3q_z)+1\,065\,982\,814\,286\,662\,929\,188\cos(2q_x-q_z)-71\,045\,518\,667\,401\,407\,385\cos(4q_x-q_z)+5\,558\,973\,502\,654\,370\,204\,522\cos(q_z)+\\
&\qquad 2\,436\,591\,489\,369\,587\,854\,102\cos(3q_z)+1\,065\,982\,814\,286\,662\,929\,188\cos(2q_x+q_z)-71\,045\,518\,667\,401\,407\,385\cos(4q_x+q_z)+1\,037\,796\,098\,184\,111\,135\,612\cos(2q_x+3q_z)+22\,425\,047\,626\,060\,301\,625\cos(4q_x+3q_z))\,\epsilon_x^3)\,\epsilon_y^4+\\
&214\,600\,(2\,488\,563\,(240\,413\,967\,594\,317\,394\,930\cos(q_x)+14\,699\,297\,937\,925\,103\,750\cos(3q_x)+1\,271\,785\,013\,595\,974\,822\cos(q_x-4q_y)+4\,399\,990\,193\,891\,960\,850\cos(3q_x-4q_y)+133\,664\,268\,006\,839\,401\,079\cos(q_x-2q_y)+23\,125\,200\,373\,177\,343\,889\cos(3q_x-2q_y)+\\
&\qquad 133\,664\,268\,006\,839\,401\,079\cos(q_x+2q_y)+23\,125\,200\,373\,177\,343\,889\cos(3q_x+2q_y)+1\,271\,785\,013\,595\,974\,822\cos(q_x+4q_y)+4\,399\,990\,193\,891\,960\,850\cos(3q_x+4q_y))\,\epsilon_y^4+\\
&\qquad 680\cos(q_x)\cos(q_y)\,(26\,377\,362\,711\,437\,143\,406\,448\cos(2q_x)+17\,(24\,350\,177\,131\,876\,798\,541\,955\cos(2\,(q_x-q_y))+103\,537\,114\,811\,431\,170\,053\,964\cos(2q_y)+24\,350\,177\,131\,876\,798\,541\,955\cos(2\,(q_x+q_y))+189\,521\,250\,296\,157\,892\,651\,058))\cos(q_z)\,\epsilon_z\,\epsilon_y^3+\\
&\qquad 190\,190\cos(q_x)\,(73\,150\,733\,551\,296\,492\,000\cos(2q_x)+394\,609\,048\,551\,392\,352\,993\cos(2\,(q_x-q_y))+6\,492\,905\,696\,156\,135\,980\,254\cos(2q_y)+394\,609\,048\,551\,392\,352\,993\cos(2\,(q_x+q_y))+\\
&\qquad 394\,609\,048\,551\,392\,352\,993\cos(2\,(q_x-q_z))+290\,989\,365\,058\,303\,439\,205\cos(2\,(q_x-q_y-q_z))+2\,718\,224\,708\,508\,735\,619\,054\cos(2\,(q_y-q_z))+290\,989\,365\,058\,303\,439\,205\cos(2\,(q_x+q_y-q_z))+6\,492\,905\,696\,156\,135\,980\,254\cos(2q_z)+\\
&\qquad 394\,609\,048\,551\,392\,352\,993\cos(2\,(q_x+q_z))+290\,989\,365\,058\,303\,439\,205\cos(2\,(q_x-q_y+q_z))+2\,718\,224\,708\,508\,735\,619\,054\cos(2\,(q_y+q_z))+290\,989\,365\,058\,303\,439\,205\cos(2\,(q_x+q_y+q_z))+5\,481\,903\,536\,908\,993\,663\,560)\,\epsilon_z^2\,\epsilon_y^2+\\
&\qquad 680\cos(q_x)\cos(q_y)\cos(q_z)\,(26\,377\,362\,711\,437\,143\,406\,448\cos(2q_x)+17\,(24\,350\,177\,131\,876\,798\,541\,955\cos(2\,(q_x-q_z))+103\,537\,114\,811\,431\,170\,053\,964\cos(2q_z)+24\,350\,177\,131\,876\,798\,541\,955\cos(2\,(q_x+q_z))+189\,521\,250\,296\,157\,892\,651\,058))\,\epsilon_z^3\,\epsilon_y+\\
&\qquad 2\,488\,563\,(240\,413\,967\,594\,317\,394\,930\cos(q_x)+14\,699\,297\,937\,925\,103\,750\cos(3q_x)+1\,271\,785\,013\,595\,974\,822\cos(q_x-4q_z)+4\,399\,990\,193\,891\,960\,850\cos(3q_x-4q_z)+133\,664\,268\,006\,839\,401\,079\cos(q_x-2q_z)+\\
&\qquad 23\,125\,200\,373\,177\,343\,889\cos(3q_x-2q_z)+133\,664\,268\,006\,839\,401\,079\cos(q_x+2q_z)+23\,125\,200\,373\,177\,343\,889\cos(3q_x+2q_z)+1\,271\,785\,013\,595\,974\,822\cos(q_x+4q_z)+4\,399\,990\,193\,891\,960\,850\cos(3q_x+4q_z))\,\epsilon_z^4)\,\epsilon_x^3+\\
&1850\,(17\,419\,941\cos(q_y)\,(2\,694\,647\,836\,666\,346\,056\,502\cos(2q_x)-79\,399\,637\,804\,625\,869\,575\cos(2\,(q_x-2q_y))+75\,315\,853\,526\,608\,759\,468\cos(2\,(q_x-q_y))-631\,264\,627\,037\,016\,027\,256\cos(2q_y)-140\,238\,254\,604\,974\,852\,450\cos(4q_y)+75\,315\,853\,526\,608\,759\,468\cos(2\,(q_x+q_y))-\\
&\qquad 79\,399\,637\,804\,625\,869\,575\cos(2\,(q_x+2q_y))+3\,516\,837\,643\,125\,107\,421\,146)\,\epsilon_y^5+2465\,(37\,237\,442\,991\,970\,252\,628\,982\,294\cos(2q_x)-312\,093\,211\,837\,576\,432\,167\,135\cos(2\,(q_x-2q_y))+17\,345\,506\,396\,856\,886\,439\,397\,076\cos(2\,(q_x-q_y))+\\
&\qquad 33\,737\,524\,641\,470\,653\,978\,366\,808\cos(2q_y)-3\,487\,168\,087\,521\,395\,361\,696\,570\cos(4q_y)+17\,345\,506\,396\,856\,886\,439\,397\,076\cos(2\,(q_x+q_y))-312\,093\,211\,837\,576\,432\,167\,135\cos(2\,(q_x+2q_y)))+64\,894\,117\,210\,323\,277\,720\,719\,330)\cos(q_z)\,\epsilon_z\,\epsilon_y^4+\\
&\qquad 12\,818\cos(q_y)\,(8\,011\,371\,695\,323\,945\,952\,255\,196\cos(2q_x)+2\,017\,084\,716\,808\,033\,597\,921\,350\cos(2\,(q_x-q_y))+3\,303\,096\,819\,059\,364\,071\,793\,132\cos(2q_y)+2\,017\,084\,716\,808\,033\,597\,921\,350\cos(2\,(q_x+q_y))+3\,564\,496\,447\,098\,991\,842\,491\,490\cos(2\,(q_x-q_z))+\\
&\qquad 769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x-q_y-q_z))+1\,529\,337\,979\,100\,810\,055\,590\,610\cos(2\,(q_y-q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x+q_y-q_z))+12\,481\,850\,044\,517\,386\,286\,724\,980\cos(2q_z)+\\
&\qquad 3\,564\,496\,447\,098\,991\,842\,491\,490\cos(2\,(q_x+q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x-q_y+q_z))+1\,529\,337\,979\,100\,810\,055\,590\,610\cos(2\,(q_y+q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x+q_y+q_z))+17\,289\,778\,089\,122\,435\,681\,309\,532)\,\epsilon_z^2\,\epsilon_y^3+\\
&\qquad 12\,818\cos(q_z)\,(8\,011\,371\,695\,323\,945\,952\,255\,196\cos(2q_x)+3\,564\,496\,447\,098\,991\,842\,491\,490\cos(2\,(q_x-q_y))+12\,481\,850\,044\,517\,386\,286\,724\,980\cos(2q_y)+3\,564\,496\,447\,098\,991\,842\,491\,490\cos(2\,(q_x+q_y))+2\,017\,084\,716\,808\,033\,597\,921\,350\cos(2\,(q_x-q_z))+\\
&\qquad 769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x-q_y-q_z))+1\,529\,337\,979\,100\,810\,055\,590\,610\cos(2\,(q_y-q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x+q_y-q_z))+3\,303\,096\,819\,059\,364\,071\,793\,132\cos(2q_z)+\\
&\qquad 2\,017\,084\,716\,808\,033\,597\,921\,350\cos(2\,(q_x+q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x-q_y+q_z))+1\,529\,337\,979\,100\,810\,055\,590\,610\cos(2\,(q_y+q_z))+769\,814\,407\,616\,991\,363\,782\,925\cos(2\,(q_x+q_y+q_z))+17\,289\,778\,089\,122\,435\,681\,309\,532)\,\epsilon_z^3\,\epsilon_y^2+\\
&2465\cos(q_y)\,(37\,237\,442\,991\,970\,252\,628\,982\,294\cos(2q_x)-312\,093\,211\,837\,576\,432\,167\,135\cos(2\,(q_x-2q_z))+17\,345\,506\,396\,856\,886\,439\,397\,076\cos(2\,(q_x-q_z))+33\,737\,524\,641\,470\,653\,978\,366\,808\cos(2q_z)-3\,487\,168\,087\,521\,395\,361\,696\,570\cos(4q_z)+\\
&\qquad 17\,345\,506\,396\,856\,886\,439\,397\,076\cos(2\,(q_x+q_z))-312\,093\,211\,837\,576\,432\,167\,135\cos(2\,(q_x+2q_z)))+64\,894\,117\,210\,323\,277\,720\,719\,330)\,\epsilon_y^4\,\epsilon_y+17\,419\,941\cos(q_z)\,(2\,694\,647\,836\,666\,346\,056\,502\cos(2q_x)-79\,399\,637\,804\,625\,869\,575\cos(2\,(q_x-2q_z))+\\
&\qquad 75\,315\,853\,526\,608\,759\,468\cos(2\,(q_x-q_z))-631\,264\,627\,037\,016\,027\,256\cos(2q_z)-140\,238\,254\,604\,974\,852\,450\cos(4q_z)+75\,315\,853\,526\,608\,759\,468\cos(2\,(q_x+q_z))-79\,399\,637\,804\,625\,869\,575\cos(2\,(q_x+2q_z))+3\,516\,837\,643\,125\,107\,421\,146)\,\epsilon_z^5)\,\epsilon_x^2-\\
&40\cos(q_x)\,(121\,939\,587\,(1\,402\,580\,864\,994\,528\,556\,478\cos(2q_y)+945\,250\,976\,433\,680\,996\,345\cos(4q_y)-5525\,(4\,261\,854\,929\,259\,466\cos(6q_y)+1\,482\,430\,507\,417\,107\,313))\,\epsilon_y^6+\\
&\qquad 22\,527\,635\,(-306\,582\,770\,556\,862\,288\,653\,010\cos(q_y)+54\,767\,957\,989\,838\,564\,683\,227\cos(3q_y)+4\,307\,453\,213\,356\,403\,211\,255\cos(5q_y))\cos(q_z)\,\epsilon_z\,\epsilon_y^5-510\,184\,675\,(4\,639\,444\,862\,944\,510\,447\,176\cos(2q_y)-838\,687\,869\,143\,105\,762\,277\cos(4q_y)-\\
&\qquad 17\,(13\,099\,400\,197\,237\,968\,510\cos(4q_y-2q_z)-148\,900\,726\,126\,399\,338\,934\cos(2\,(q_y-q_z))-556\,208\,371\,442\,325\,201\,600\cos(2q_z)-148\,900\,726\,126\,399\,338\,934\cos(2\,(q_y+q_z))+13\,099\,400\,197\,237\,968\,510\cos(2\,(2q_y+q_z))-817\,160\,160\,332\,733\,313\,005))\,\epsilon_z^2\,\epsilon_y^4-\\
&\qquad 23\,713\,300\cos(q_y)\cos(q_z)\,(77\,415\,446\,480\,467\,820\,945\,874\cos(2q_y)+41\,326\,990\,955\,428\,667\,706\,525\cos(2\,(q_y-q_z))+77\,415\,446\,480\,467\,820\,945\,874\cos(2q_z)+41\,326\,990\,955\,428\,667\,706\,525\cos(2\,(q_y+q_z))+720\,165\,127\,414\,798\,218\,506\,546)\,\epsilon_z^3\,\epsilon_y^3-\\
&\qquad 510\,184\,675\,(9\,455\,542\,314\,519\,528\,427\,200\cos(2q_y)-222\,689\,803\,353\,045\,464\,670\cos(2\,(q_y-2q_z))+2\,531\,312\,344\,148\,788\,761\,878\cos(2\,(q_y-q_z))+4\,639\,444\,862\,944\,510\,447\,176\cos(2q_z)-838\,687\,869\,143\,105\,762\,277\cos(4q_z)+2\,531\,312\,344\,148\,788\,761\,878\cos(2\,(q_y+q_z))-\\
&\qquad 222\,689\,803\,353\,045\,464\,670\cos(2\,(q_y+2q_z))+13\,891\,722\,725\,656\,466\,321\,085)\,\epsilon_z^4\,\epsilon_y^2+22\,527\,635\cos(q_y)\,(-306\,582\,770\,556\,862\,288\,653\,010\cos(q_z)+54\,767\,957\,989\,838\,564\,683\,227\cos(3q_z)+4\,307\,453\,213\,356\,403\,211\,255\cos(5q_z))\,\epsilon_z^5\,\epsilon_y+\\
&\qquad 121\,939\,587\,(1\,402\,580\,864\,994\,528\,556\,478\cos(2q_z)+945\,250\,976\,433\,680\,996\,345\cos(4q_z)-5525\,(4\,261\,854\,929\,259\,466\cos(6q_z)+1\,482\,430\,507\,417\,107\,313))\,\epsilon_z^6)\,\epsilon_x+\\
&29\,(28\,168\,044\,597\,(827\,808\,374\,798\,014\,805\cos(q_y)-407\,511\,682\,759\,452\,297\cos(3q_y)+32\,682\,412\,580\,801\,125\cos(5q_y)-335\,214\,948\,722\,225\cos(7q_y))\,\epsilon_y^7+79\,513\,556\,850\,(-3\,199\,521\,946\,356\,060\,169\cos(2q_y)-426\,269\,439\,903\,081\,070\cos(4q_y)+22\,157\,903\,009\,071\,225\cos(6q_y)+8\,822\,380\,144\,549\,761\,790)\\
&\qquad \cos(q_z)\,\epsilon_z\,\epsilon_y^6-7030\cos(q_y)\,(107\,935\,258\,530\,325\,509\,440\,917\,704\cos(2q_y)+1\,988\,748\,601\,768\,714\,806\,578\,250\cos(4q_y)+3\,488\,726\,031\,704\,474\,627\,341\,275\cos(4q_y-2q_z)+34\,791\,218\,319\,990\,708\,715\,324\,380\cos(2\,(q_y-q_z))-\\
&\qquad 302\,741\,637\,586\,991\,729\,892\,000\,430\cos(2q_z)+34\,791\,218\,319\,990\,708\,715\,324\,380\cos(2\,(q_y+q_z))+3\,488\,726\,031\,704\,474\,627\,341\,275\cos(2\,(2q_y+q_z))-272\,208\,306\,739\,742\,605\,716\,514\,002)\,\epsilon_z^2\,\epsilon_y^5+\\
&\qquad 78\,625\,(3\,699\,848\,237\,896\,519\,803\,002\,532\cos(2q_y-3q_z)+198\,728\,133\,319\,062\,035\,450\,685\cos(4q_y-3q_z)+9\,188\,393\,949\,480\,738\,286\,320\,572\cos(2q_y-q_z)-1\,558\,299\,818\,480\,711\,534\,986\,485\cos(4q_y-q_z)+55\,806\,125\,520\,765\,647\,027\,674\,674\cos(q_z)+\\
&\qquad 3\,508\,054\,663\,308\,705\,034\,350\,462\cos(3q_z)+9\,188\,393\,949\,480\,738\,286\,320\,572\cos(2q_y+q_z)-1\,558\,299\,818\,480\,711\,534\,986\,485\cos(4q_y+q_z)+3\,699\,848\,237\,896\,519\,803\,002\,532\cos(2q_y+3q_z))\,\epsilon_z^3\,\epsilon_y^4+\\
&\qquad 78\,625\,(55\,806\,125\,520\,765\,647\,027\,674\,674\cos(q_y)+3\,508\,054\,663\,308\,705\,034\,350\,462\cos(3q_y)-1\,558\,299\,818\,480\,711\,534\,986\,485\cos(q_y-4q_z)+198\,728\,133\,319\,062\,035\,450\,685\cos(3q_y-4q_z)+9\,188\,393\,949\,480\,738\,286\,320\,572\cos(q_y-2q_z)+\\
&\qquad 3\,699\,848\,237\,896\,519\,803\,002\,532\cos(3q_y-2q_z)+9\,188\,393\,949\,480\,738\,286\,320\,572\cos(q_y+2q_z)+3\,699\,848\,237\,896\,519\,803\,002\,532\cos(3q_y+2q_z)-1\,558\,299\,818\,480\,711\,534\,986\,485\cos(q_y+4q_z)+198\,728\,133\,319\,062\,035\,450\,685\cos(3q_y+4q_z))\,\epsilon_z^4\,\epsilon_y^3-\\
&\qquad 7030\cos(q_z)\,(3\,(1\,162\,908\,677\,234\,824\,875\,780\,425\cos(2\,(q_y-2q_z))+11\,597\,072\,773\,330\,236\,238\,441\,460\cos(2\,(q_y-q_z))+35\,978\,419\,510\,108\,503\,146\,972\,568\cos(2q_z)+662\,916\,200\,589\,571\,602\,192\,750\cos(4q_z)+11\,597\,072\,773\,330\,236\,238\,441\,460\cos(2\,(q_y+q_z))+\\
&\qquad\qquad 1\,162\,908\,677\,234\,824\,875\,780\,425\cos(2\,(q_y+2q_z))-90\,736\,102\,246\,580\,868\,572\,171\,334)-302\,741\,637\,586\,991\,729\,892\,000\,430\cos(2\,q_y))\,\epsilon_z^5\,\epsilon_y^2+79\,513\,556\,850\cos(q_y)\\
&\qquad (-3\,199\,521\,946\,356\,060\,169\cos(2q_z)-426\,269\,439\,903\,081\,070\cos(4q_z)+22\,157\,903\,009\,071\,225\cos(6q_z)+8\,822\,380\,144\,549\,761\,790)\,\epsilon_z^6\,\epsilon_y+28\,168\,044\,597\,(827\,808\,374\,798\,014\,805\cos(q_z)-407\,511\,682\,759\,452\,297\cos(3q_z)+32\,682\,412\,580\,801\,125\cos(5q_z)-335\,214\,948\,722\,225\cos(7q_z))\,\epsilon_z^7))
\end{aligned}
$$

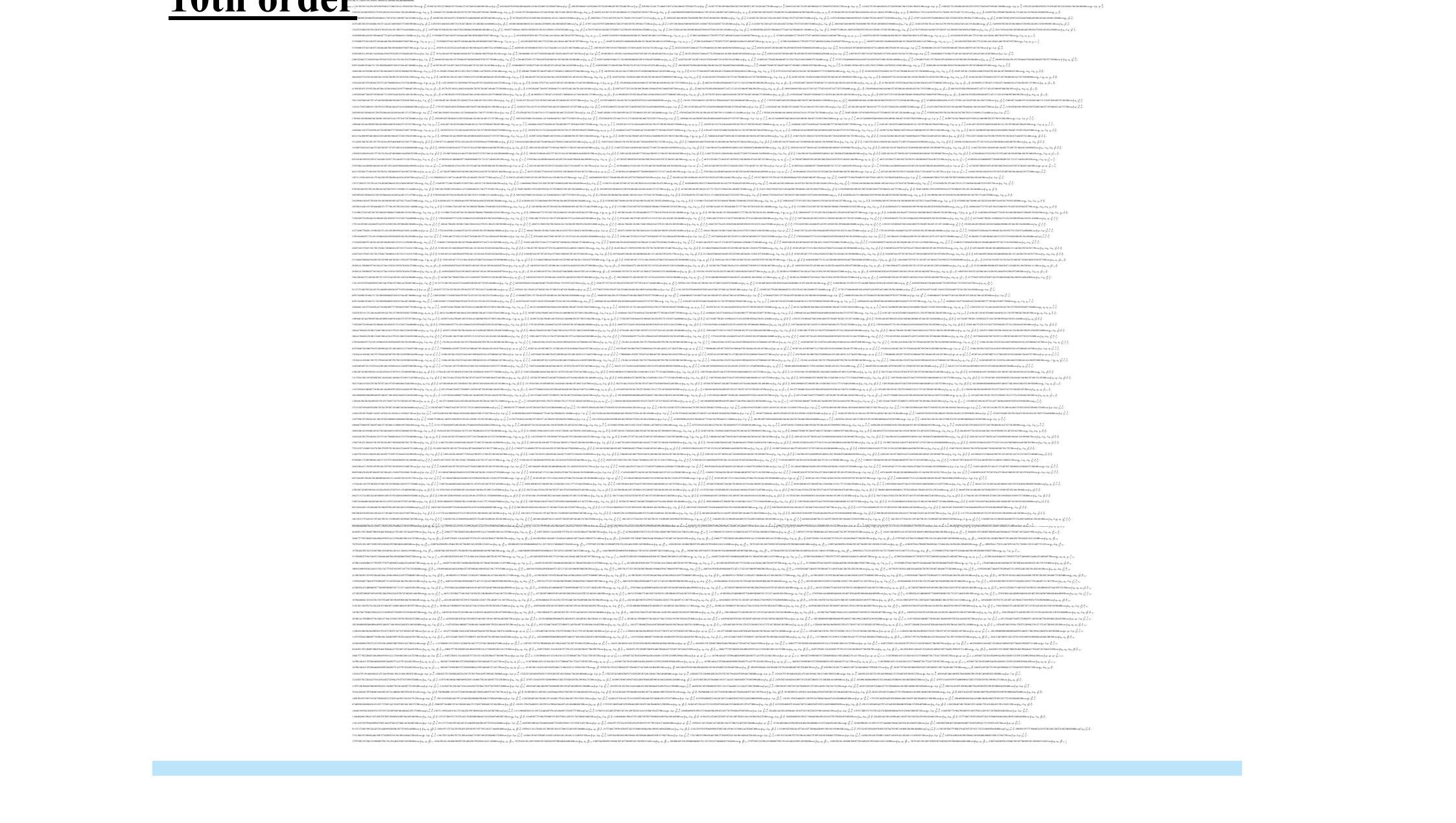
4.1 Expressions

9th order

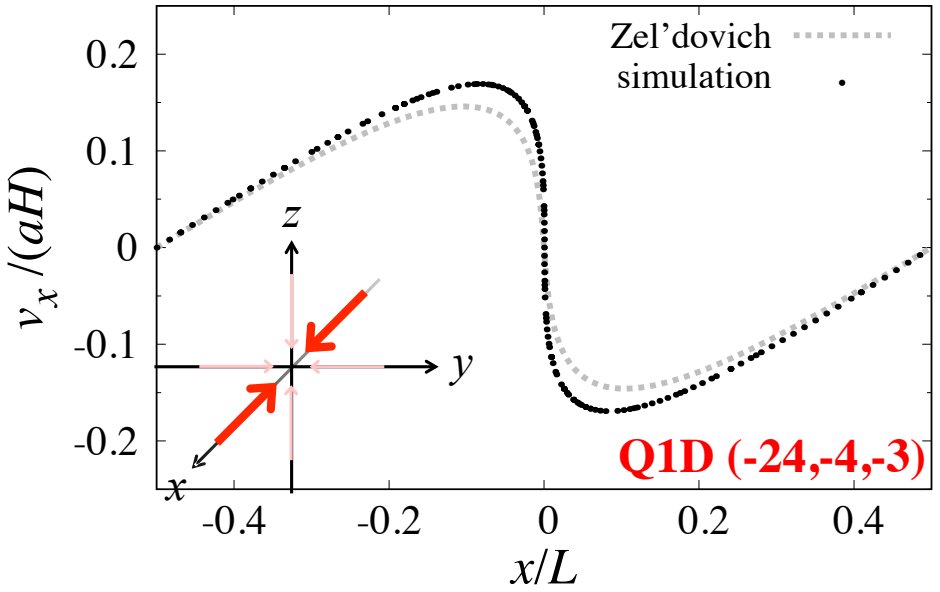


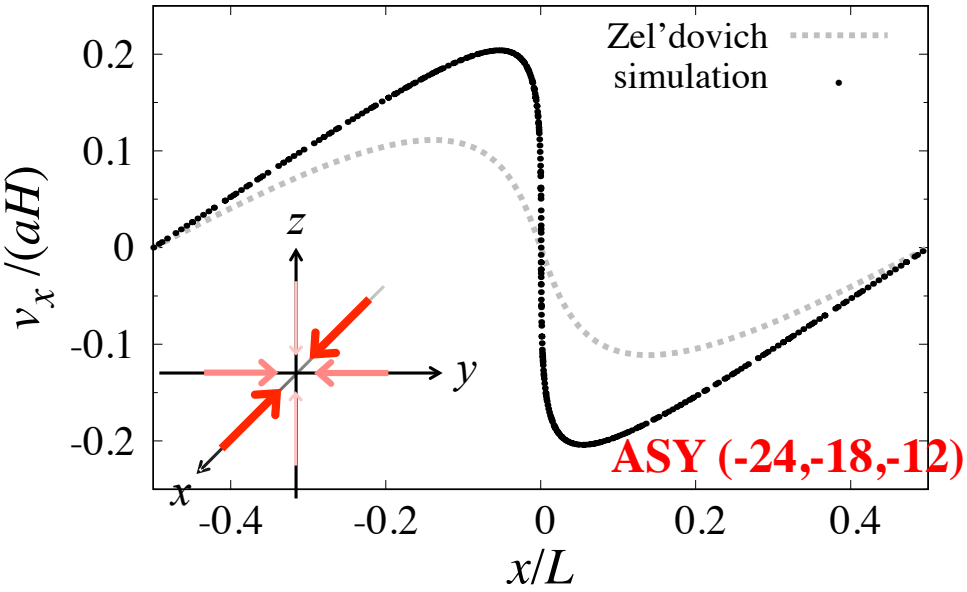
4.1 Expressions

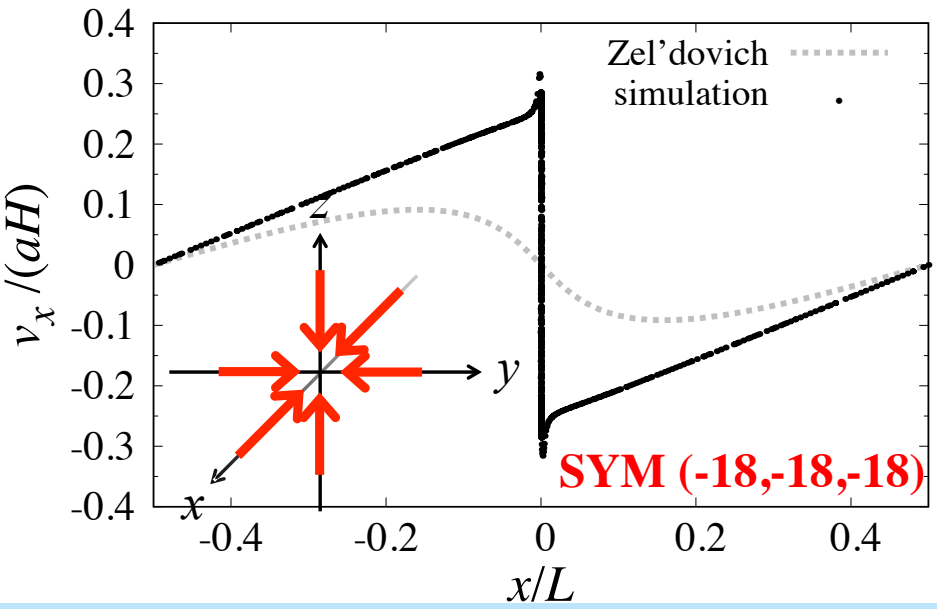
10th order



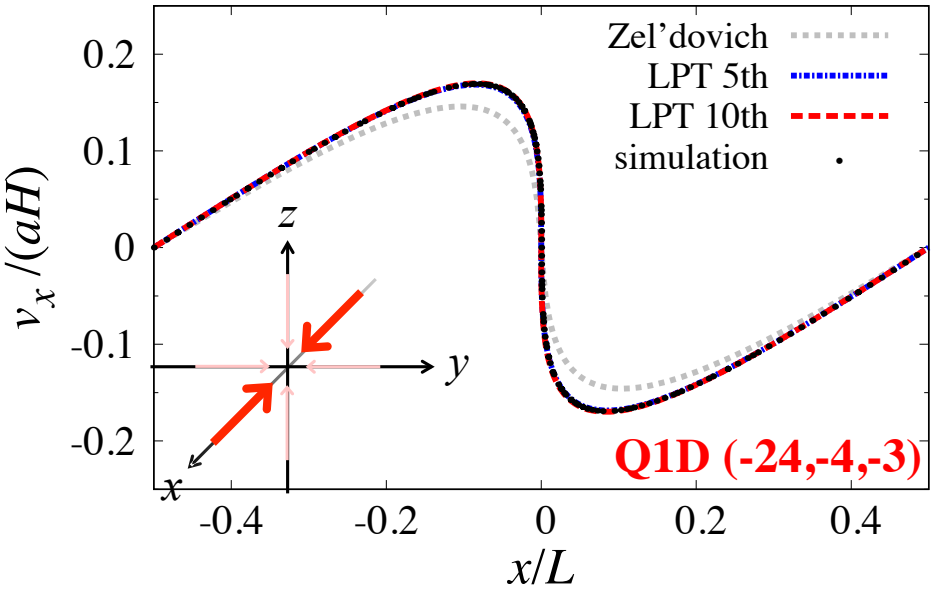
4.2 Zel'dovich approximation at shell-crossing

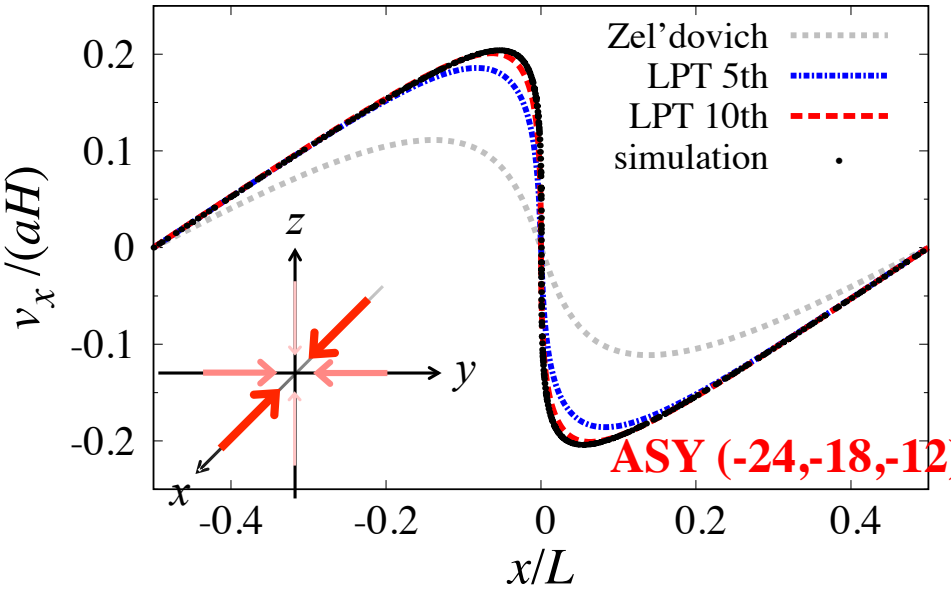


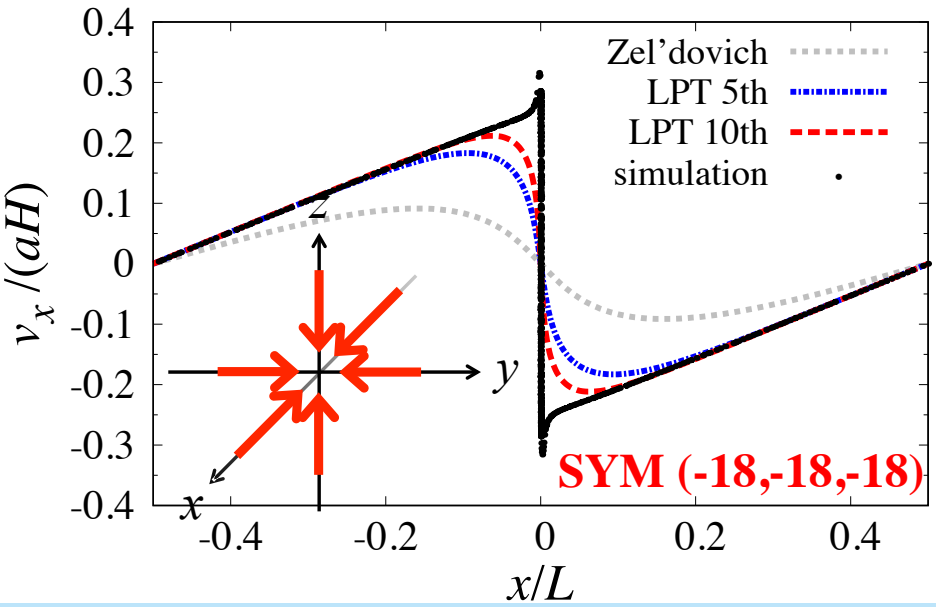




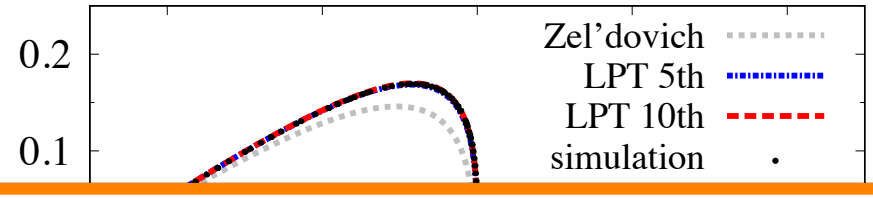
4.3 5th and 10th LPT at shell-crossing

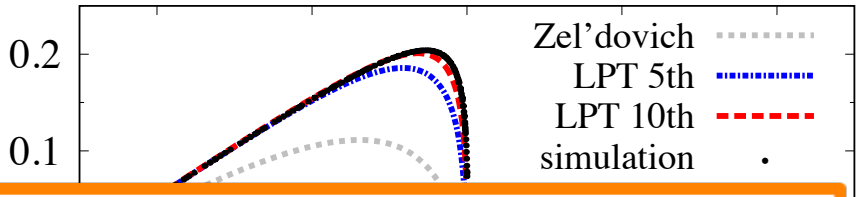






4.4 5th and 10th LPT at shell-crossing



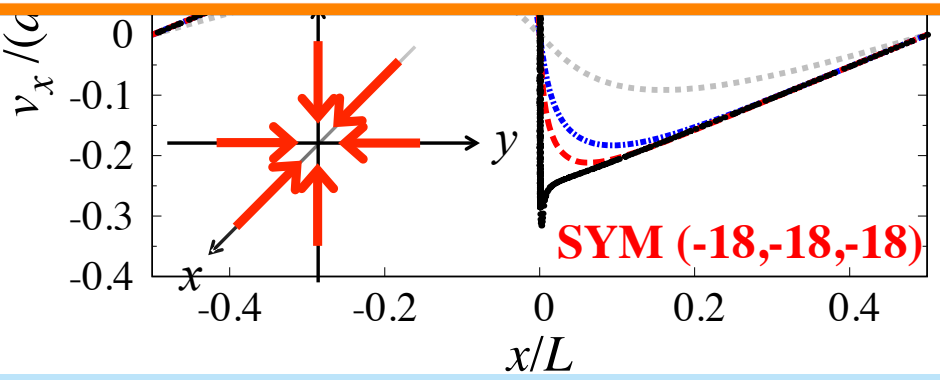


We found the fitting formula $x_i(\boldsymbol{q}, t, n)$ as a function of order n.

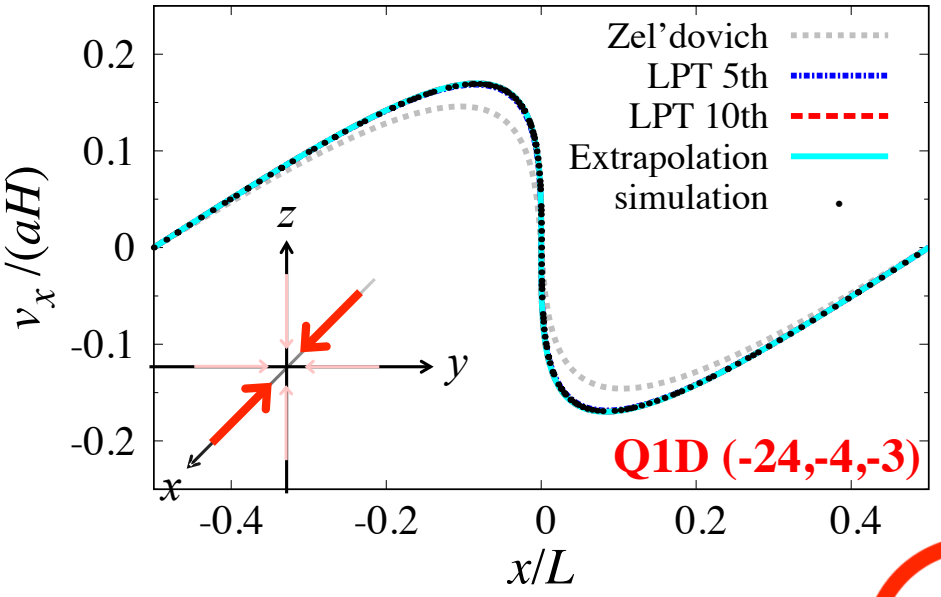
$$x_i(\boldsymbol{q}, t, n) = q_i + \sum_{a=1}^{n} \Psi_i^{(a)}(\boldsymbol{q}, t)$$

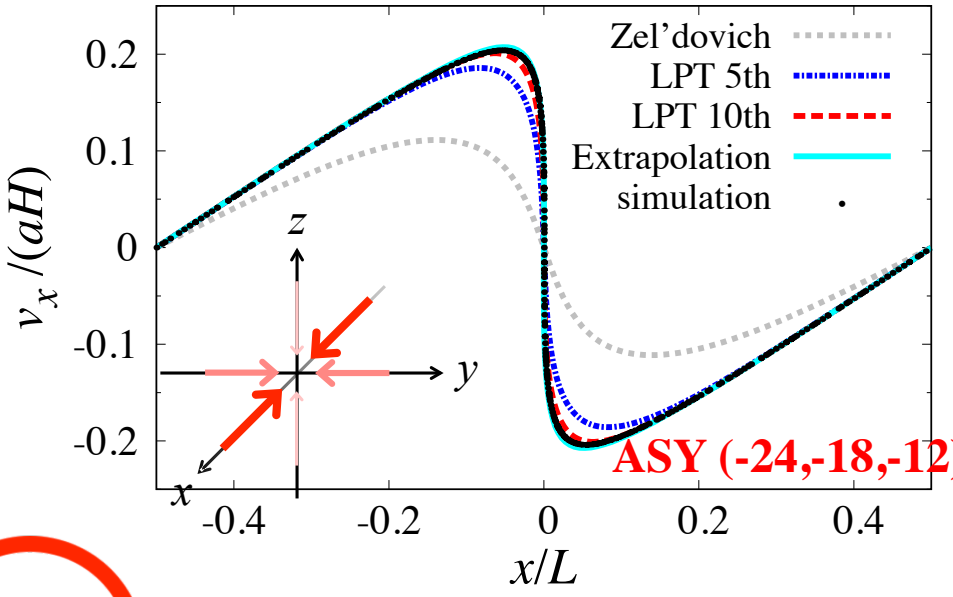
$$= a_i(\boldsymbol{q}, t) + \frac{1}{b_i(\boldsymbol{q}, t) + c_i(\boldsymbol{q}, t) \exp\left[d_i(\boldsymbol{q}, t) n^{e_i(\boldsymbol{q}, t)}\right]}$$

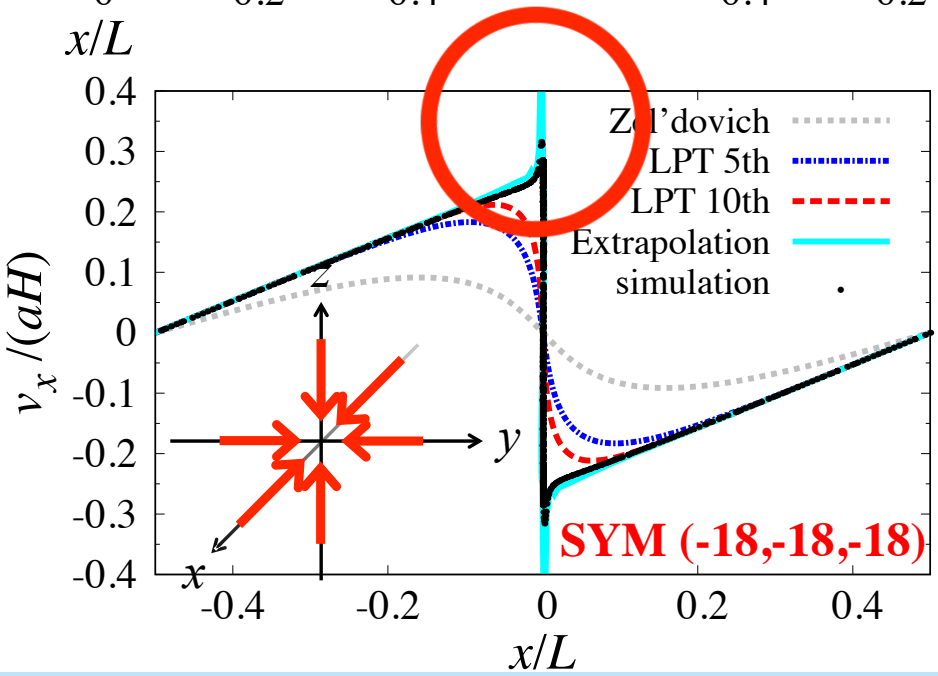
$n \to \infty$: "Effective" infinite-order LPT



4.5 Extrapolation at shell-crossing

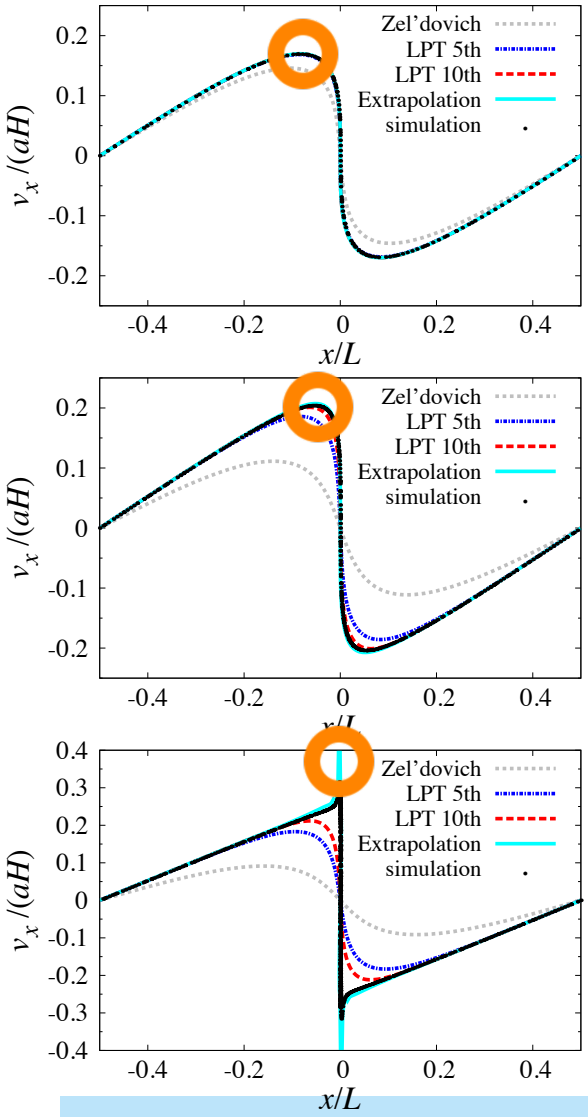


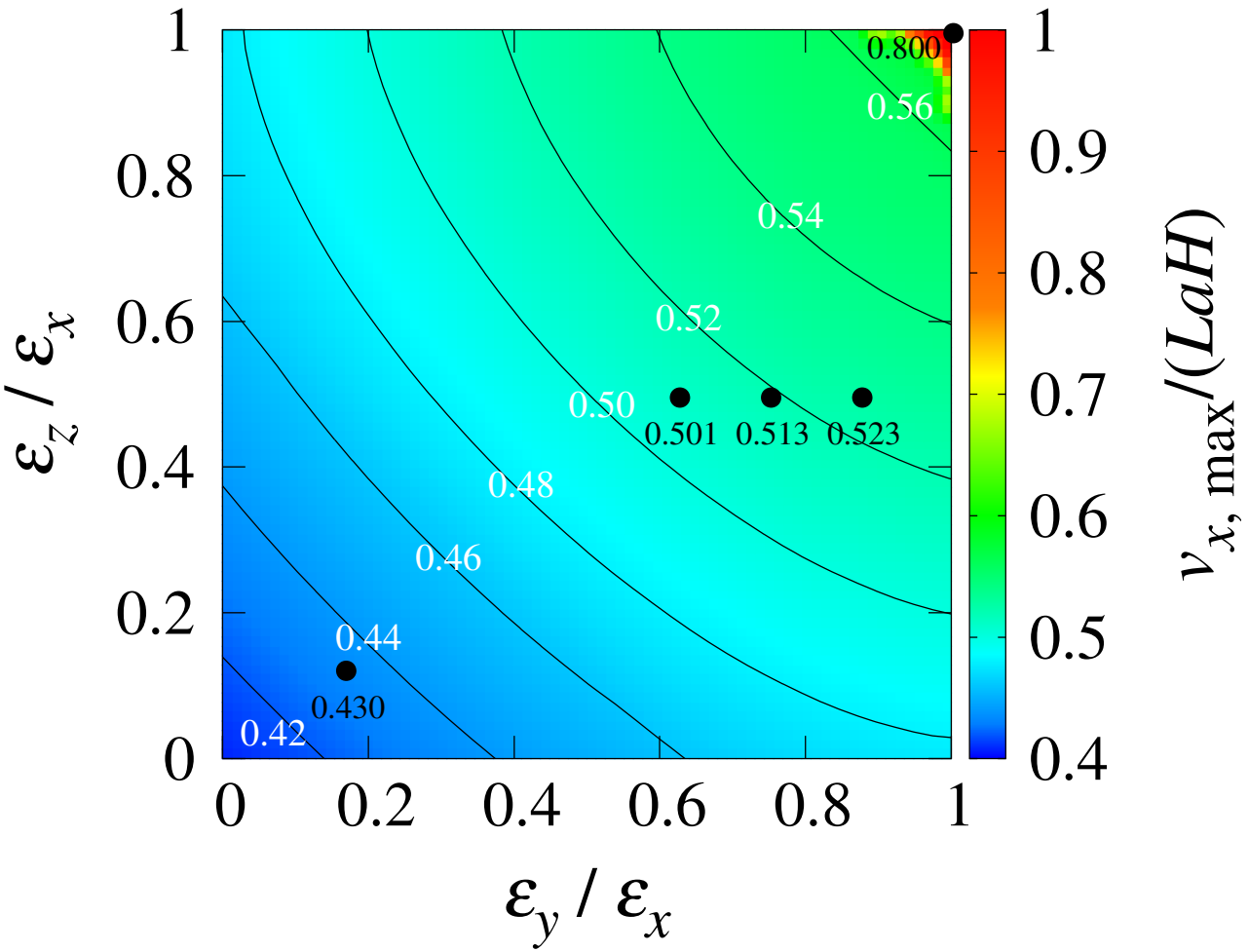




4.6 Exploring parameter space@shell-crossing

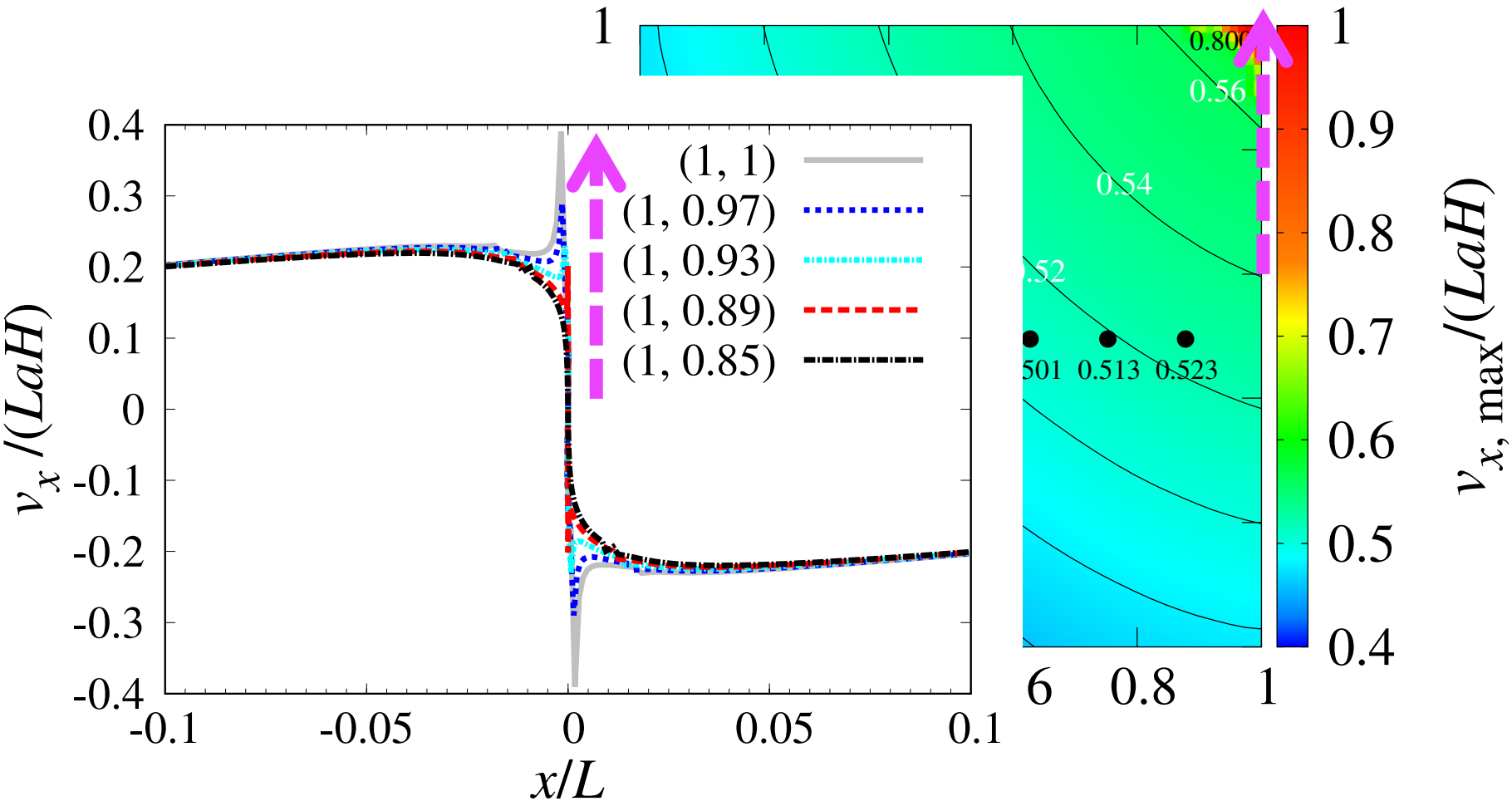
Focusing on maximum velocity as a function of $(\varepsilon_y/\varepsilon_x, \varepsilon_z/\varepsilon_x)$.





4.6 Exploring parameter space@shell-crossing

Focusing on maximum velocity as a function of $(\varepsilon_y/\varepsilon_x, \varepsilon_z/\varepsilon_x)$.



5. Summary

We consider

- ✓ primordial dark matter halos seeded by three sine waves.

We perform

- ✓ **10th order** LPT and **beyond**
- ✓ **state-of-the-art high-resolution Vlasov-Poisson simulation**

We found

- ✓ except for the triaxial symmetric configuration, **shell-crossing is generally expected to produce a planar singularity**
- ✓ **2D → 3D: spiky feature appears**

We expect

- hint/first step to understand the nature of DM halo profile
- 3D spiky feature (rare region) ~ related to rare halos?