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Axion as a cold dark matter candidate: proof to fully nonlinear order

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Axions and the strong *CP* problem

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Current upper bounds on the neutron electric dipole moment constrain the physically observable quantum chromodynamic (QCD) vacuum angle $|\bar{\theta}| \lesssim 10^{-11}$. Since QCD explains a great deal of experimental data from the 100 MeV to the TeV scale, it is desirable to explain this smallness of $|\bar{\theta}|$ in the QCD framework; this is the strong *CP* problem. There now exist two plausible solutions to this problem, one of which leads to the existence of a very light axion.

axions may constitute a significant fraction of the dark matter of the universe.

Scalar field:

$$T_{ab} = \phi_{,a}\phi_{,b} - \left(\frac{1}{2}\phi^{;c}\phi_{,c} + V\right) g_{ab}$$

$$\phi^{;c}{}_{c} = V_{,\phi}$$

← **Klein-Gordon equation: Relativistic Schrödinger equation for boson**

Axion:

$$V = \frac{1}{2}m^2\phi^2$$

Axion as a CDM candidate, proofs:

Background order:	Abbott, Sikivie (1983) Dine, Fischler (1983) Preskill, Wise, Wilczek (1983)
Newtonian:	Khlopov, Malomed, Zeldovich (1985)
Linear perturbation order:	Nambu, Sasaki (1990) **Zero-shear gauge** Ratra (1991) **Synchronous gauge** JH (1997) **Uniform-curvature gauge** Sikivie, Yang (2009) **Zero-shear gauge** JH, Noh (2009) **Axion comoving gauge (ACG)**
Extreme low-mass:	Fuzzy dark matter, huge industry!
Nonlinear order:	Noh, Park, JH (2013, 2015) **ACG**
Fully nonlinear and exact:	(2017) **ACG**

Fully NL & Exact Pert. Theory

JH, Noh, MN **433** 3472 (2013)
JH, Noh, Park, MN **461** 3239 (2016)
Gong, JH, Noh, Wu, Yoo, JCAP **10** 027 (2017)

Convention: (Bardeen 1988)

Decomposition, possible to NL order

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a^2 \left(\beta_{,i} + B_i^{(v)}\right) d\eta dx^i$$
$$+ a^2 \left[(1 + 2\varphi) g_{ij}^{(3)} + 2\gamma_{,i|j} + C_{i|j}^{(v)} + C_{j|i}^{(v)} + 2C_{ij}^{(t)}\right] dx^i dx^j,$$

No TT-pert!

$$\chi \equiv a\beta + a^2\dot{\gamma}, \quad \Psi_i^{(v)} \equiv B_i^{(v)} + a\dot{C}_i^{(v)},$$

Spatial gauge condition

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{u}_a\widetilde{u}_b + \widetilde{g}_{ab}\right) + \widetilde{\pi}_{ab}, \quad \widetilde{u}_i \equiv a\gamma\frac{v_i}{c} \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{1}{1+2\varphi}\frac{v^2}{c^2}}}$$

$$\widetilde{\mu} \equiv \mu + \delta\mu \equiv \mu(1 + \delta), \quad \widetilde{p} \equiv p + \delta p, \quad v_i \equiv -v_{,i} + v_i^{(v)}$$

Spatial gauge: $\gamma \equiv 0 \equiv C_i^{(v)},$

$$\chi_i \equiv \chi_{,i} + a\Psi_i^{(v)} = a\left(\beta_{,i} + B_i^{(v)}\right)$$

Temporal gauge still not taken yet!

⇒ Complete spatial gauge fixing.
Remaining variables are spatially gauge-invariant to fully NL order! ∴ Lose no generality!

(JH, Noh 2013)

Metric convention:

$$\widetilde{g}_{00} = -a^2 \left(1 + 2\alpha\right), \qquad \widetilde{g}_{0i} = -a\chi_i, \qquad \widetilde{g}_{ij} = a^2 \left(1 + 2\varphi\right) g^{(3)}_{ij}.$$

Inverse metric:

$$\widetilde{g}^{00} = -\frac{1}{a^2} \frac{1 + 2\varphi}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2},$$

$$\widetilde{g}^{0i} = -\frac{1}{a^2} \frac{\chi^i / a}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2},$$

$$\widetilde{g}^{ij} = \frac{1}{a^2(1 + 2\varphi)} \left(g^{(3)ij} - \frac{\chi^i \chi^j / a^2}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2} \right).$$

Exact!

Using the ADM and the covariant formalisms the rest are simple algebra. We do not even need the connection!

→ **Fully Nonlinear Perturbation Equations without taking temporal gauge condition**

Noh, JH, MNRAS (2013); Noh, JCAP **07** (2014) 037

With TT perturbation:

TT

$$\widetilde{g}_{00} = -a^2\left(1+2\alpha\right), \quad \widetilde{g}_{0i} = -a\chi_i, \quad \widetilde{g}_{ij} = a^2\left[\left(1+2\varphi\right)\delta_{ij} + 2h_{ij}\right].$$

raised and lowered using δ_{ij}

T

$$+\,2\gamma_{,ij} + 2C^{(v)}_{(i,j)} \equiv 0$$

Spatial Gauge taken

=spatial Harmonic to 1PN

$$\widetilde{g}^{00} = -\frac{1}{a^2\mathcal{N}^2}, \quad \widetilde{g}^{0i} = -\frac{\delta^{ij}+H^{ij}}{a^3\mathcal{N}^2(1+2\varphi+I)}\chi_j,$$

$$\widetilde{g}^{ij} = \frac{1}{a^2(1+2\varphi+I)}\left(\delta^{ij}+H^{ij} - \frac{(\delta^{ik}+H^{ik})(\delta^{j\ell}+H^{j\ell})}{a^2\mathcal{N}^2(1+2\varphi+I)}\chi_k\chi_\ell\right).$$

$$H^{ij} \equiv -2\frac{(1+2\varphi)h^{ij} - 2h^{ik}h^j_k}{(1+2\varphi)^2 - 2h^{k\ell}h_{k\ell}}, \quad I \equiv \frac{8}{3}\frac{h_{k\ell}h^k_m h^{\ell m}}{(1+2\varphi)^2 - 2h^{k\ell}h_{k\ell}}$$

$$N = a\sqrt{1+2\alpha+\frac{\delta^{ij}+H^{ij}}{a^2(1+2\varphi+I)}\chi_i\chi_j} \equiv a\mathcal{N}$$

Gong, JH, Noh, Wu, Yoo, JCAP (2017)

Without any assumption:

T → $C^{(v)}_{(i,j)}$ TT → C_{ij}

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2 B_i d\eta dx^i + a^2\left[(1+2\varphi)\delta_{ij} + 2\gamma_{,ij} + 2C^{(v)}_{(i,j)} + 2C_{ij}\right]dx^i dx^j$$

$$B_i = \beta_{,i} + B^{(v)}_i \qquad \chi_i \equiv a\left(B_i + a\dot{\gamma}_{,i} + a\dot{C}^{(v)}_i\right) \qquad Y_{ij} \equiv \gamma_{,ij} + C^{(v)}_{(i,j)} + C_{ij}$$

$$\det\left[h_{ij}\right] = a^6\left[(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{kl}Y_{kl}\right]$$

ADM intrinsic curvature

$$\times\left\{1 + 2\underbrace{\left[\varphi + \frac{2}{3}\frac{(\Delta\gamma)^3 - 3(\Delta\gamma)Y^{kl}Y_{kl} + 2Y_{kl}Y^k{}_m Y^{lm}}{(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{pq}Y_{pq}}\right]}_{\equiv\widehat{\varphi}}\right\},$$

$$\epsilon^i{}_{lm}\epsilon^j{}_{pq}h_{pl}h_{qm} = 2a^4\left[(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{kl}Y_{kl}\right]$$

$$\times\left[\delta^{ij}\underbrace{-2\frac{(1+2\varphi+2\Delta\gamma)Y^{ij} - 2Y^{ik}Y^j{}_k}{(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{lm}Y_{lm}}}_{\equiv H^{ij}}\right],$$

$$g^{00} = -\frac{1}{a^2\mathcal{N}^2}$$

$$g^{0i} = -\frac{\delta^{ij} + H^{ij}}{a^2\mathcal{N}^2(1+2\widehat{\varphi})}\frac{\chi_j}{a},$$

All spatial indices raised and lowered by δ_{ij}

$$g^{ij} = \frac{1}{a^2(1+2\widehat{\varphi})}\left[\delta^{ij} + H^{ij} - \frac{(\delta^{ik}+H^{ik})(\delta^{jl}+H^{jl})}{a^2\mathcal{N}^2(1+2\widehat{\varphi})}\chi_k\chi_l\right].$$

$$N = a\sqrt{1 + 2\alpha + \frac{\delta^{ij}+H^{ij}}{a^2(1+2\widehat{\varphi})}\chi_i\chi_j} \equiv a\mathcal{N}$$

Gong, JH, Noh, Wu, Yoo, JCAP (2017)

Temporal gauge (slicing, hypersurface):

comoving gauge : $v \equiv 0,$

zero−shear gauge : $\chi \equiv 0,$ **Longitudinal, Newtonian, Poisson, …**

uniform−curvature gauge : $\varphi \equiv 0,$

uniform−expansion gauge : $\kappa \equiv 0,$ ← **Perturbed trace of extrinsic curvature, K**

K = - 3H + κ

~Maximal Slicing (K ≡ 0)

uniform−density gauge : $\delta \equiv 0,$

synchronous gauge : $\alpha \equiv 0.$ → **Remnant gauge mode**

Applicable to fully NL orders!

⇒

Except for synchronous gauge, complete gauge fixing. Remaining variables are gauge-invariant to fully NL order!

Zero-pressure irrotational fluid

Comoving gauge

Linear-order: (Lifshitz 1946, synchronous gauge, comoving gauge)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Second-order: (Noh, JH 2004, comoving gauge)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}),$$

Relativistic/Newtonian correspondence to second order.
This equation is valid to fully nonlinear order in Newtonian theory.

Third-order: (JH, Noh 2005, comoving gauge)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$

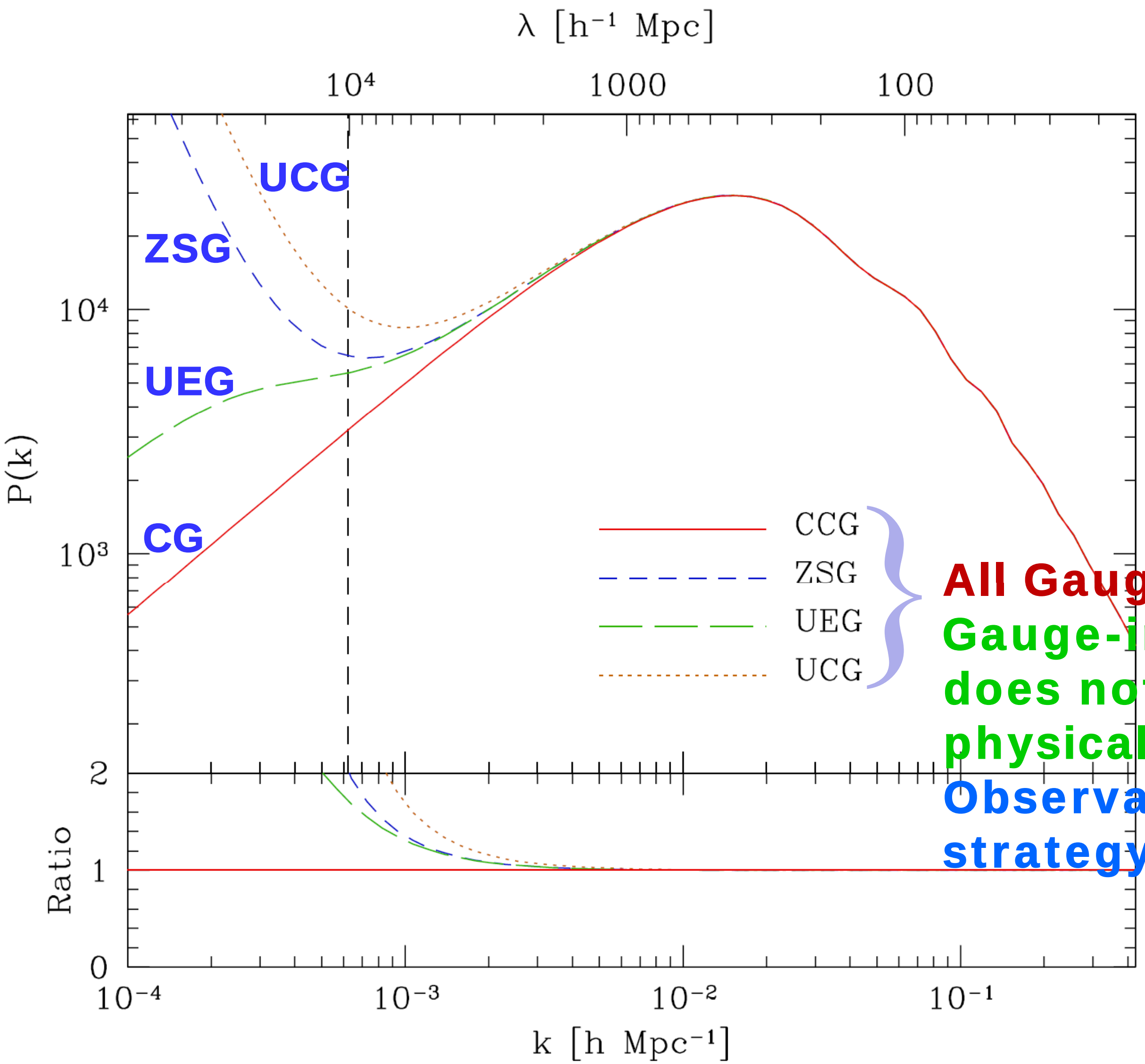
Pure relativistic correction appearing from third order. All terms involve $\varphi=\varphi_v$

$$+\frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$

$$+\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u}\cdot\nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},$$

$$X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla\left(\nabla\varphi\right) + \mathbf{u}\Delta\varphi\right].$$

Baryonic matter power spectrum in the CDM model: linear order

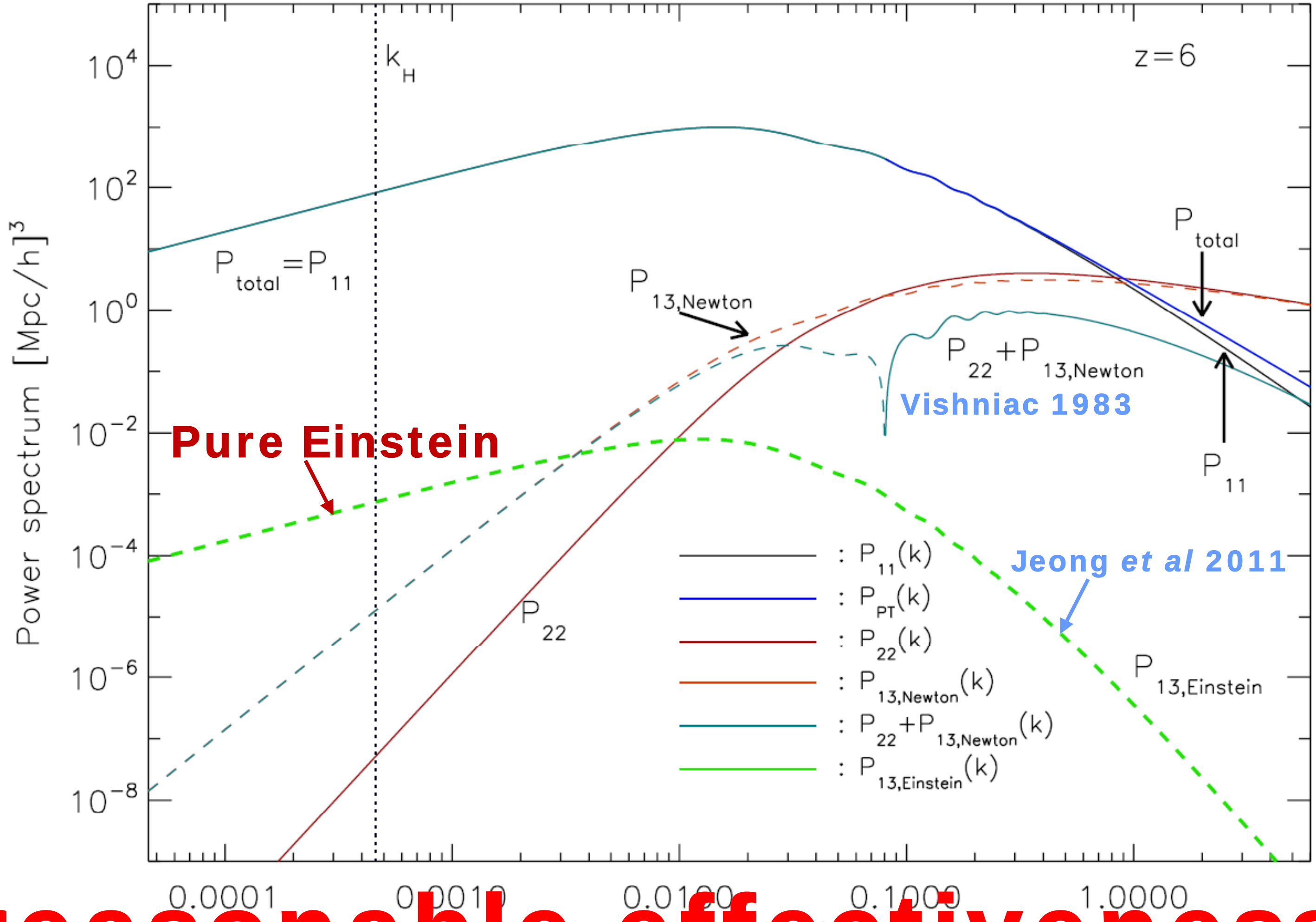


All Gauge-invariant!

Gauge-invariance does not guarantee physical observable!

Observational strategy may tell.

Leading Nonlinear Density Power-spectrum in the Comoving gauge



Unreasonable effectiveness of Newton's gravity in cosmology!

(Jeong, Gong, Noh, JH 2011)

Zero-pressure irrotational fluid in the comoving gauge (no TT)

Covariant energy-conservation:

$$\dot{\delta} - \kappa - \delta\kappa + \frac{1}{a^2}\chi^{,i}\delta_{,i} = \frac{2\varphi\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

Trace of ADM propagation (Raychaudhury equation):

$$\dot{\kappa} + 2H\kappa - 4\pi G\delta\mu - \frac{1}{3}\kappa^2 + \frac{1}{a^2}\chi^{,i}\kappa_{,i} - \frac{1}{a^4}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right] = \frac{2\varphi\chi^{,i}\kappa_{,i}}{a^2(1+2\varphi)} - \frac{4\varphi(1+\varphi)}{a^4(1+2\varphi)^2}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right]$$
$$+ \frac{2}{a^4(1+2\varphi)^3}\left\{\frac{2}{3}(\Delta\chi)\chi^{,i}\varphi_{,i} - 2\chi^{,ij}\chi_{,i}\varphi_{,j} + \frac{1}{1+2\varphi}\left[\frac{1}{3}(\chi^{,i}\varphi_{,i})^2 + \chi^{,i}\chi_{,i}\varphi^{,j}\varphi_{,j}\right]\right\} - \frac{\hbar^2\Delta}{2m^2a^4}\frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

ADM momentum constraint:

$$\left(\kappa + \frac{\Delta}{a^2}\chi\right)_{,i} = \frac{2\varphi\Delta\chi_{,i}}{a^2(1+2\varphi)} + \frac{1}{a^2(1+2\varphi)^2}\left[2(\Delta\chi)\varphi_{,i} + \frac{1}{2}\chi^{,k}\varphi_{,ik} - \chi_{,ik}\varphi^{,k} + \frac{3}{2}\chi_{,i}\Delta\varphi - \frac{3}{2}\frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + \frac{1}{3}\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right].$$

Newtonian

RHS = pure Einstein's gravity corrections, starting from the third order, all involving φ

Curvature perturbation in the comoving gauge $\mathcal{R}$

Definition of kappa + ADM momentum constraint:

$$[\ln(1+2\varphi)]^{\cdot}_{,i} = \frac{1}{a^2(1+2\varphi)^2}\left[\chi^{,k}\varphi_{,ik} + \chi_{,i}\Delta\varphi - \frac{1}{1+2\varphi}(\chi_{,i}\varphi_{,k} + 3\chi_{,k}\varphi_{,i})\varphi^{,k}\right],$$

Identify: $\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}$

Perturbed part of the trace of extrinsic curvature

(JH, Noh 2013)

Axion!

- Madelung, E., Z. Phy **40** 322 (1927)
- Khlopov, Malomed, Zeldovich, MN **215** 575 (1985)
- Bose-Einstein condensation
- Noh, JH, Park, ApJ (2017)

Axion

Axion comoving gauge

Background:

Strictly ignore: $$\frac{H_0}{m} = 2.133 \times 10^{-28} h \left(\frac{m}{10^{-5}\ \text{eV}}\right)^{-1}$$

Ansatz: $$\phi(t) = \phi_+(t)\sin(mt) + \phi_-(t)\cos(mt)$$

(Ratra 1991)

EOM: $$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$$

$$\Rightarrow \phi(t) = a^{-3/2}[\phi_{+0}\sin(mt) + \phi_{-0}\cos(mt)]$$

Fluid quantities:

$$\mu = \frac{1}{2}\langle\dot{\phi}^2 + m^2\phi^2\rangle = \frac{1}{2}m^2 a^{-3}(\phi_{+0}^2 + \phi_{-0}^2),$$

$$p = \frac{1}{2}\langle\dot{\phi}^2 - m^2\phi^2\rangle = 0,$$

$$\langle f(t)\rangle \equiv \frac{m}{2\pi}\int_0^{2\pi/m} f(t')dt'$$

$\therefore$ **Pressureless fluid!**

Linear order:

Ansatz: $$\delta\phi(k,t) = \delta\phi_+(k,t)\sin(mt) + \delta\phi_-(k,t)\cos(mt).$$

Fluids:
$$\delta\mu = \langle \dot\phi\delta\dot\phi - \dot\phi^2\alpha + m^2\phi\delta\phi \rangle,$$
$$\delta p = \langle \dot\phi\delta\dot\phi - \dot\phi^2\alpha - m^2\phi\delta\phi \rangle,$$
$$\frac{a}{k}(\mu + p)v = \langle \dot\phi\delta\phi \rangle, \qquad \sigma = 0.$$

Anisotropic stress

EOM:
$$\delta\ddot\phi + 3H\delta\dot\phi + \frac{k^2}{a^2}\delta\phi + V_{,\phi\phi}\delta\phi = \dot\phi(\kappa + \dot\alpha) + (2\ddot\phi + 3H\dot\phi)\alpha.$$

Axion-comoving gauge: $(\mu_a + p_a)v_a \equiv 0.$

(JH, Noh 2009)

Derivation in Axion comoving gauge:

Energy conservation & Raychaudhury equations:

$$\dot{\delta} = \kappa$$

$$\dot{\kappa} + 2H\kappa = 4\pi G\varrho\delta + \frac{\Delta}{a^2}\frac{\delta p}{\varrho}$$

EOM
$T^c_{i;c}$
$T^c_{0;c}$

$$\delta p = -\frac{1}{4}\frac{\Delta}{m^2a^2}\delta\varrho$$

$$\Rightarrow \quad \ddot{\delta} + 2H\dot{\delta} - \left(4\pi G\varrho - \frac{1}{4}\frac{\Delta^2}{m^2a^4}\right)\delta = 0$$

Valid in ALL cosmological scales

(JH, Noh2009)

Axion Jeans scale:

$$\ddot{\delta} + 2H\dot{\delta} - \left(4\pi G\varrho - \frac{1}{4}\frac{k^4}{m^2 a^4}\right)\delta = 0$$

Sound speed: $c_s \equiv \sqrt{\frac{\delta p}{\delta \mu}} = \frac{1}{2}\frac{k}{ma}$

Jeans scale: (Khlopov, Malomed, Zeldovich 1985; Nambu, Sasaki 1990)

$$\lambda_J \equiv \frac{2\pi a}{k_J} \equiv \left(\frac{\pi^3}{G\varrho m^2}\right)^{1/4}$$

$$= 50h^{-1/2}\left(\frac{m}{10^{-5}eV}\right)^{-1/2}\mathrm{AU}$$

$$= 2.4h^{-1/2}\left(\frac{m}{10^{-25}eV}\right)^{-1/2}\mathrm{Mpc}$$

∴ **CDM in ALL cosmological scales for QCD axion**
Fuzzy dark matter for extremely low-mass axion

The Astrophysical Journal, 846:1 (6pp), 2017 September 1
https://doi.org/10.3847/1538-4357/aa8366

CrossMark

Axion as a Cold Dark Matter Candidate: Proof to Fully Nonlinear Order

Hyerim Noh[1], Jai-chan Hwang[2], and Chan-Gyung Park[3]

Fully nonlinear and exact:

Perturbed lapse:
$$\delta\mathcal{N} = \frac{\hbar^2}{2m^2a^2c^2}\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}$$

Perturbed pressure:
$$\delta p = -\frac{\hbar^2}{4m^2a^2}\Delta\varrho$$

Anisotropic stress:
$$\Pi_{ij} = \frac{\hbar^2}{4m^2a^2}\frac{1}{\varrho}\left(\varrho_{,i}\varrho_{,j} - \frac{1}{3}\delta_{ij}\varrho^{,k}\varrho_{,k}\right)$$

Weak gravity limit:

Energy conservation & Raychaudhury equations:

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} + \frac{1}{a}\nabla \cdot (\delta \mathbf{u}) = 0$$

$$\frac{1}{a}\nabla \cdot \left(\dot{\mathbf{u}} + H\mathbf{u}\right) + 4\pi G\varrho\delta + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$$
$$= \frac{\hbar^2 \Delta}{2m^2 a^4} \frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

$$\Rightarrow \quad \ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta + \frac{1}{a^2}\left[a\nabla \cdot (\delta \mathbf{u})\right]^{\cdot} - \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$$
$$= -\frac{\hbar^2 \Delta}{2m^2 a^4} \frac{\Delta\sqrt{1+\delta}}{\sqrt{1+\delta}}$$

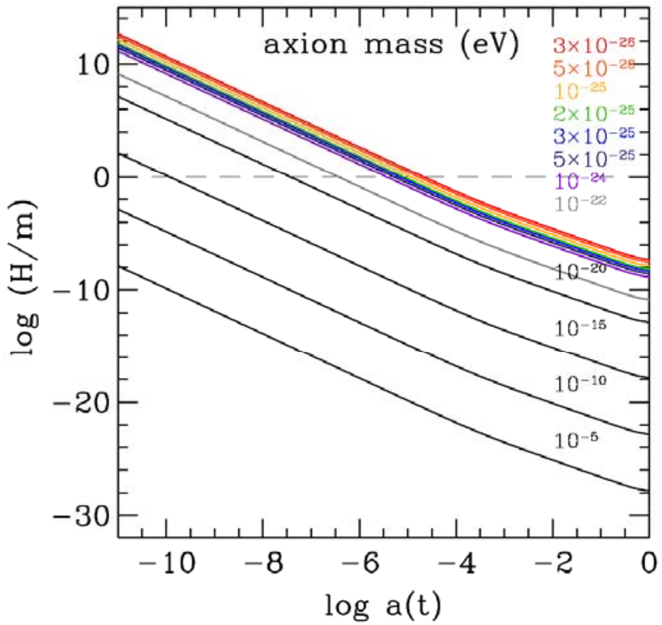
Zero-pressure, irrotational, no GW in Axion comoving gauge
Exact!

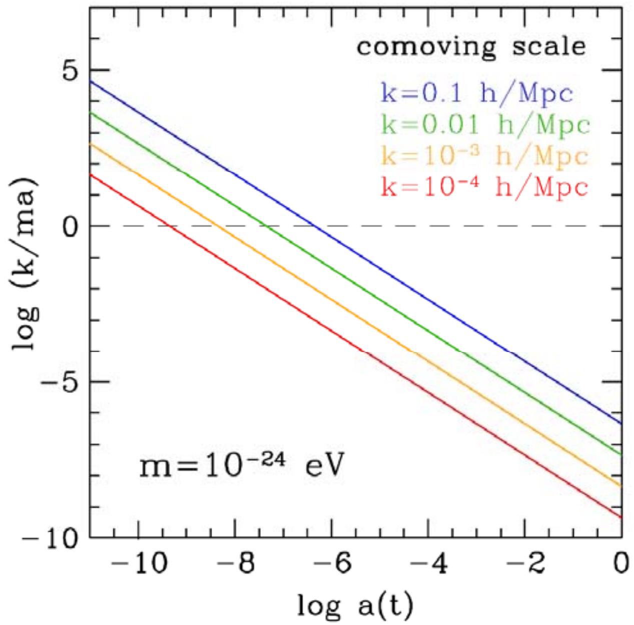
Low-mass Axion

Fuzzy dark matter

Low-mass Axion: (Park, JH, Noh 2012)

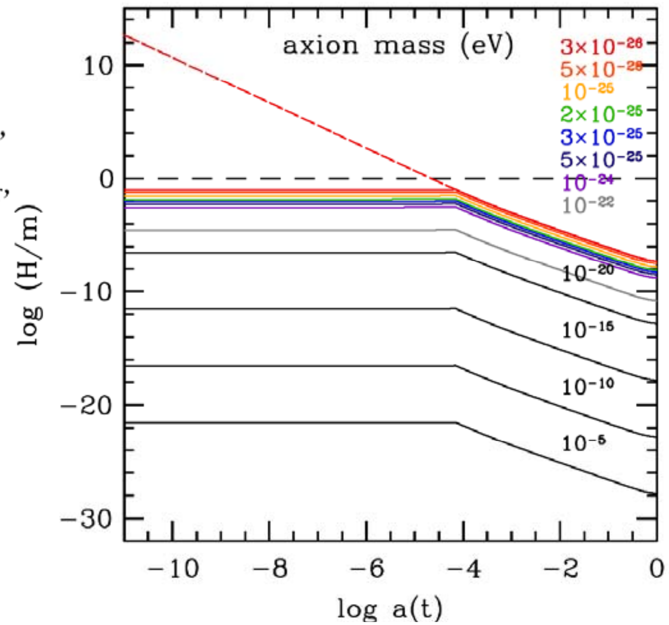
Constant mass:

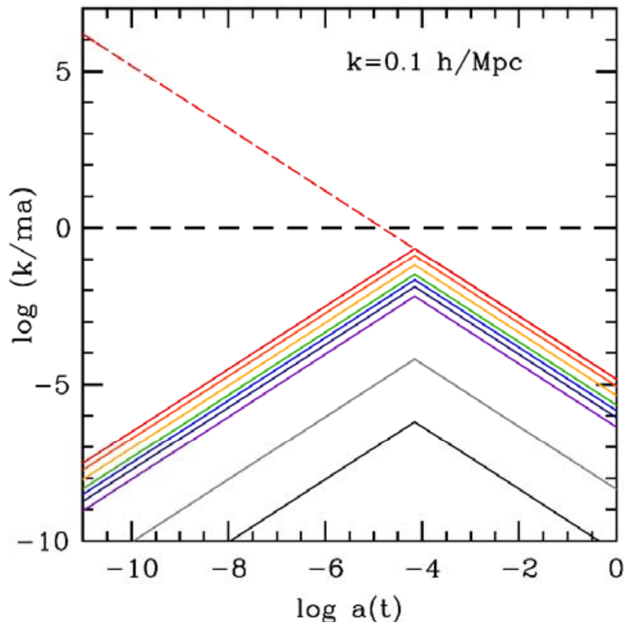




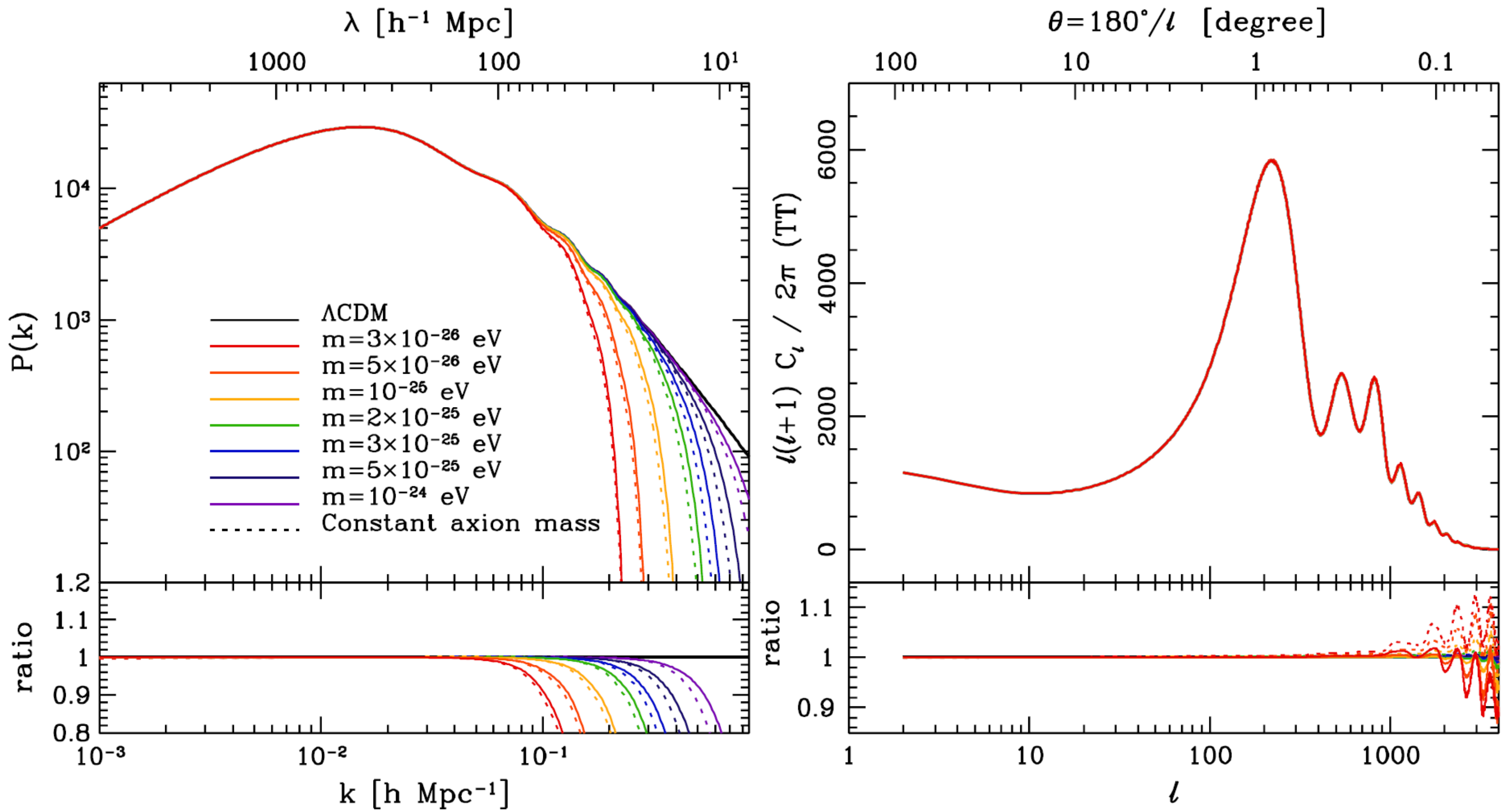
Evolving mass:

$$m(a) = \begin{cases} m_0 (a_{tr}/a)^2 & a < a_{tr}, \\ m_0 & a \geq a_{tr}, \end{cases}$$





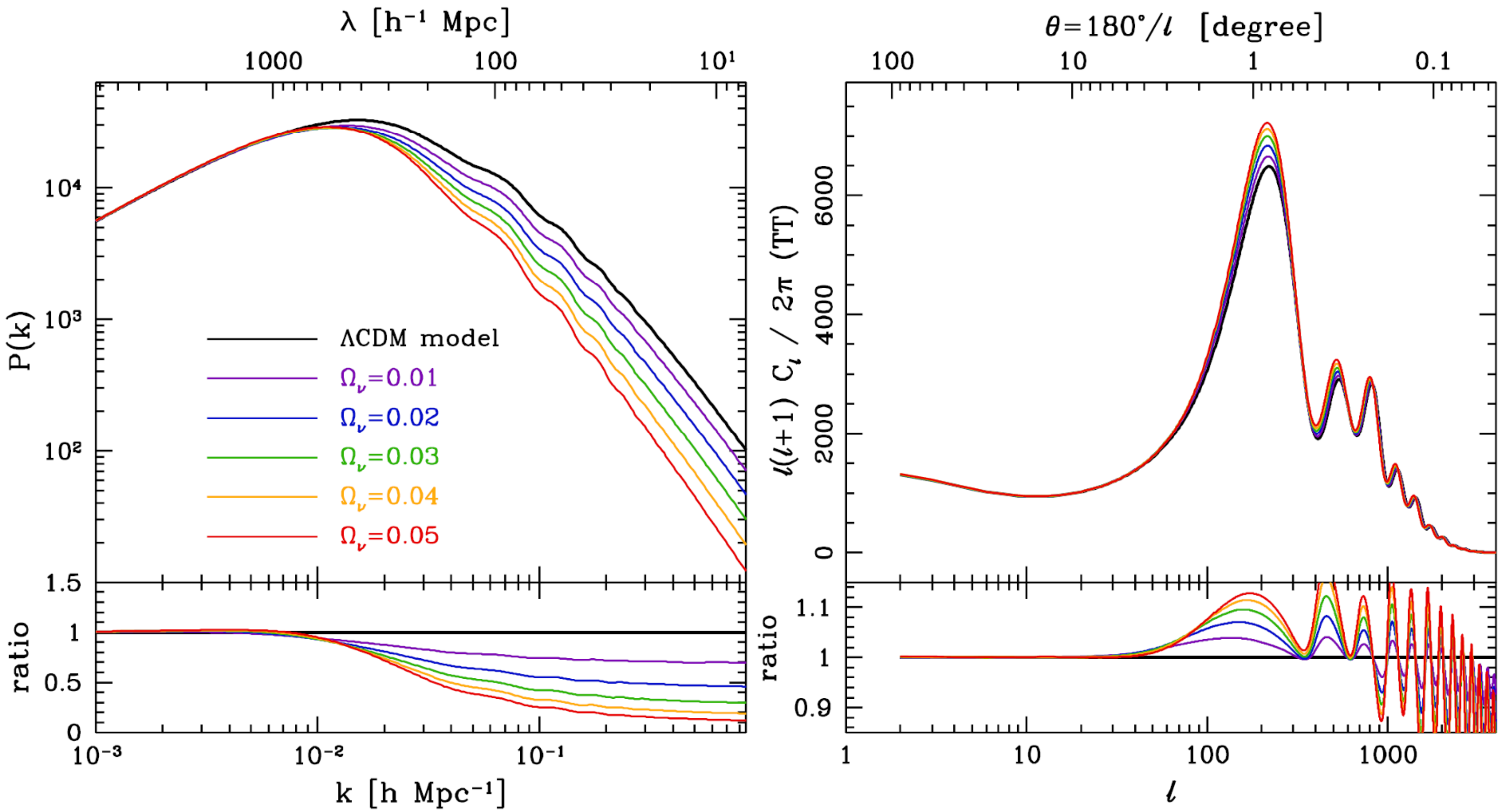
Low-mass axion as a fuzzy CDM: (Park, JH, Noh 2012)



If the dark matter is composed of FDM, most observations favor a particle mass $\gtrsim 10^{-22}$ eV

Hui, Ostriker, Tremaine, Witten (2017)

Neutrino as a HDM: (Park, JH, Noh 2012)



mass ranges from $m_\nu = 0.154$ eV ($\Omega_{\nu 0} = 0.01$; red) to 0.769 eV ($\Omega_{\nu 0} = 0.05$; violet curves), with a relation $(\Omega_{\nu 0} + \Omega_{c0})h^2 = 0.1123$. Black curves represent the power spectrum of the fiducial ΛCDM model with massless neutrinos. The curves in the bottom

Quantum Nature

Madelung's QM interpretation:

Quantentheorie in hydrodynamischer Form.

Von **E. Madelung** in Frankfurt a. M.

(Eingegangen am 25. Oktober 1926.)

$$\Delta\psi - \frac{8\pi^2 m}{h^2} U\psi - i\frac{4\pi m}{h}\frac{\partial\psi}{\partial t} = 0. \qquad (2)$$

$$\psi = \alpha e^{i\beta} \qquad \varphi = -\frac{\beta h}{2\pi m} \qquad \mathfrak{u} = \operatorname{grad}\varphi$$

Identify: $\mathbf{u} \equiv \nabla u \equiv \nabla\varphi \equiv -\frac{\hbar}{m}\nabla\beta \qquad \alpha \equiv \sqrt{\varrho}$

Im → Continuity eq.:

$$\operatorname{div}(\alpha^2 \operatorname{grad}\varphi) + \frac{\partial\alpha^2}{\partial t} = 0. \qquad (4')$$

Re → Mom-conserv.:

$$\frac{\partial\mathfrak{u}}{\partial t} + \frac{1}{2}\operatorname{grad} u^2 = \frac{d\mathfrak{u}}{dt} = -\frac{\operatorname{grad} U}{m} + \operatorname{grad}\frac{\Delta\alpha}{\alpha}\frac{h^2}{8\pi^2 m^2}. \qquad (3'')$$

Quantum pressure

$$\psi = \sqrt{\varrho}e^{-imu/\hbar}$$

Jeans scale due to uncertainty principle:

$$\lambda_{J_a} \equiv \frac{2\pi a}{k_{J_a}} = \sqrt{\frac{\pi\hbar}{m}\sqrt{\frac{\pi}{G\varrho}}} \sim 5.4 \times 10^{14}\mathrm{cm}\sqrt{\frac{10^{-5}\mathrm{eV}}{mh}}$$

de Broglie wavelength

$$\Leftarrow \quad \lambda \sim \frac{h}{mv_g} \sim \frac{h}{m\lambda/t_g} \sim \frac{h}{m\lambda\sqrt{G\varrho}}$$

$$v_g \sim \frac{\lambda}{t_g} \qquad t_g \sim \frac{1}{\sqrt{G\varrho}}$$

(Hu, Barkana, Gruzinov 2000)